Realized Volatility Options

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Let the underlying process Y be a positive semimartingale, and let $X_t := \log(Y_t/Y_0)$.

Define realized variance to be $[X]$, where $[\cdot]$ denotes quadratic variation (but see section 5).

Define a *realized variance option* on Y with variance strike Q and expiry T to pay

$$
\begin{aligned}
&([X]_T - Q)^+ \quad \text{in the case of a realized variance call,}\\
&(Q - [X]_T)^+ \quad \text{in the case of a realized variance put,}
\end{aligned}
$$

and define a *realized volatility option* on Y with volatility strike $Q^{1/2}$ and expiry T to pay

$$
\begin{aligned}
&([X]_T^{1/2} - Q^{1/2})^+ \quad \text{in the case of a realized volatility call,}\\
&(Q^{1/2} - [X]_T^{1/2})^+ \quad \text{in the case of a realized volatility put.}
\end{aligned}
$$

We will in some places restrict attention to puts, by put-call parity: for realized variance options, a long-call short-put combination pays $[X]_T - Q$, equal to a Q-strike variance swap; and for realized volatility options, a long-call short-put combination pays $[X]_T^{1/2} - Q^{1/2}$, equal to a $Q^{1/2}$-strike volatility swap.

Unlike variance swaps [EQF07/024, EQF07/045], which admit exact model-free (assuming only continuity of Y) hedging and pricing in terms of Europeans, variance and volatility options have a range of values consistent with given prices of Europeans. With no further assumptions, there exist sub/super-replication strategies and lower/upper pricing bounds (section 4). Under an independence condition, there exist exact pricing formulas in terms of Europeans (section 2). Under specific models, there exist exact pricing formulas in terms of model parameters (section 1).

Unless otherwise noted, all prices are denominated in units of a T-maturity discount bond. The results apply to dollar-denominated prices, provided that interest rates vary deterministically, because if Y' is a dollar-denominated share price and Y is that share's bond-denominated price, then $\log Y - \log Y'$ has finite variation, so $[\log Y] = [\log Y']$.

Expectations $\mathbb{E}$ will be with respect to martingale measure $\mathbb{P}$.

Transform analysis

Some of the methods surveyed here (in particular, sections 1 and 2.1) will price variance/volatility options by pricing a continuum of payoffs of the form $e^{z[X]_T}$. Transform analysis relates the former to the latter. This presentation follows [5].

Assume that the continuous payoff function $h : \mathbb{R} \to \mathbb{R}$ satisfies

$$\int_{-\infty}^{\infty} e^{-\alpha q} h(q) \mathrm{d}q < \infty,$$

and that $\mathbb{E}e^{\alpha[X]_T} < \infty$ for some $\alpha \in \mathbb{R}$. For all $z \in \alpha + \mathbb{R}i := \{z \in \mathbb{C} : \operatorname{Re} z = \alpha\}$, define the bilateral Laplace transform

$$H(z) := \int_{-\infty}^{\infty} e^{-zq} h(q) \mathrm{d}q.$$

If $|H|$ is integrable along $\alpha + \mathbb{R}i$ for some $\alpha \leq 0$, then by Bromwich and Fubini, the $h([X]_T)$ payoff has price

$$\mathbb{E}h([X]_T) = \frac{1}{2\pi i} \int_{\alpha-\infty i}^{\alpha+\infty i} H(z) \mathbb{E}e^{z[X]_T} \mathrm{d}z. \tag{0.1}$$

For a *variance put*, let $h(q) = (Q - q)^+$. Then for all $\alpha < 0$, formula (0.1) holds with

$$H(z) = \frac{e^{-Qz}}{z^2}. \tag{0.2}$$

For a *volatility put*, let $h(q) = (\sqrt{Q} - \sqrt{q^+})^+$. Then for all $\alpha < 0$, formula (0.1) holds with

$$H(z) = -\frac{\sqrt{\pi} \operatorname{Erf}(\sqrt{zQ})}{2z^{3/2}}. \tag{0.3}$$

To price variance and volatility calls by put-call parity, we have the variance swap value

$$\mathbb{E}[X]_T = \left.\frac{\partial}{\partial z}\right|_{z=0} \mathbb{E}e^{z[X]_T}, \tag{0.4}$$

and the volatility swap value

$$\mathbb{E}[X]_T^{1/2} = \frac{1}{2\sqrt{\pi}} \int_0^{\infty} \frac{1 - \mathbb{E}e^{-z[X]_T}}{z^{3/2}} \mathrm{d}z, \tag{0.5}$$

if $\mathbb{E}e^{z[X]_T}$ is analytic in a neighborhood of $z = 0$.

1 Pricing by modeling the underlying process

Under Heston and under Lévy models, we give formulas for the transform $\mathbb{E}e^{z[X]_T}$, where $\operatorname{Re} z \leq 0$. Hence the formula (0.1) prices the variance put and volatility put, using (0.2) and (0.3) respectively.

1.1 Example: Heston dynamics

Under the Heston model for instantaneous variance,

$$\mathrm{d}V_t = (a - \kappa V_t)\mathrm{d}t + \beta\sqrt{V_t}\mathrm{d}W_t, \tag{1.1}$$

and the transform of $[X]_T = \int_0^T V_t \mathrm{d}t$ is

$$\mathbb{E}e^{z[X]_T} = e^{A(z)+B(z)V_0}, \tag{1.2}$$

where

$$
\begin{aligned}
A(z) &:= \frac{a}{\beta^2}\left[(\kappa-\gamma)T - 2\log\left(1+\frac{\kappa-\gamma}{2\gamma}(1-e^{-\gamma T})\right)\right] \\
B(z) &:= \frac{2z(e^{\gamma T}-1)}{2\gamma+(\gamma+\kappa)(e^{\gamma T}-1)} \qquad \gamma := \sqrt{\kappa^2-2\beta^2 z},
\end{aligned}
\tag{1.3}
$$

by [6].

1.2 Example: Lévy dynamics

If X is a Lévy process with Gaussian variance σ^2 and Lévy measure ν, then $[X]$ has transform

$$
\mathbb{E}e^{z[X]_T} = \exp\left(T\frac{\sigma^2 z^2}{2} + T\int_{\mathbb{R}}\big(e^{zx^2}-1\big)\nu(\mathrm{d}x)\right). \tag{1.4}
$$

For variance option pricing under pure-jump processes with independent increments, but without assuming stationary increments, see [2].

2 Pricing by use of Europeans, under an independence condition

In this section, let Y be a share price that follows general stochastic volatility dynamics

$$
\mathrm{d}Y_t = \sigma_t Y_t \mathrm{d}W_t \tag{2.1}
$$

where σ and the Brownian motion W are independent. Although all three subsections use this assumption, the schemes in sections 2.1 and 2.2 are immunized, to first order, against violations of the independence condition.

2.1 Pricing via transform

The transform of $[X]_T = \int_0^T \sigma_t^2 \mathrm{d}t$ satisfies, by [5],

$$
\mathbb{E}e^{z[X]_t} = \mathbb{E}\Big(\theta_+(Y_T/Y_0)^{1/2+\sqrt{1/4+2z}} + \theta_-(Y_T/Y_0)^{1/2-\sqrt{1/4+2z}}\Big), \tag{2.2}
$$

provided that the expectations are finite. Here $\theta_\pm := (1 \mp 1/\sqrt{1+8z})/2$. The right-hand side of (2.2) is in principle observable from T-expiry Europeans, which allows variance/volatility put option pricing by the formulas (0.1)-(0.3). In this context, (0.4) can be replaced by the log-contract value $-2\mathbb{E}X_T$, and (0.5) can be replaced by the synthetic volatility swap value [EQF08/006].

Moreover, [5] shows that (2.2) still holds approximately in the presence of correlation between σ and W, in the sense that the right-hand side is constructed to have zero sensitivity to first-order correlation effects.

2.2 Pricing and hedging via uniform or L^2 payoff approximation

For continuous payoffs $h : [0,\infty) \to \mathbb{R}$ with finite limit at ∞, such as the variance put or volatility put, consider an nth-order approximation to $h(q)$

$$A_n(q) := a_{n,n}e^{-cnq} + a_{n,n-1}e^{-c(n-1)q} + \cdots + a_{n,0}. \tag{2.3}$$

where $c > 0$ is an arbitrary constant.

To choose A by uniform approximation, the $a_{n,k}$ may be determined as the coefficients of the nth Bernstein polynomial approximation to the function $x \mapsto h(-(1/c)\log x)$ on $[0,1]$.

Then [5] shows that

$$\mathbb{E}h([X]_T) = \lim_{n\to\infty} \mathbb{E}\sum_{k=0}^{n} a_{n,k}\Big(\theta_+(Y_T/Y_0)^{1/2+\sqrt{1/4-2ck}} + \theta_-(Y_T/Y_0)^{1/2-\sqrt{1/4-2ck}}\Big), \tag{2.4}$$

where $\theta_\pm := (1 \mp 1/\sqrt{1-8ck})/2$. The right-hand side of (2.4) is in principle observable from T-expiry Europeans, and is moreover designed to have zero sensitivity to first-order correlation effects.

Alternatively, to choose A by L^2 approximation, the $a_{n,k}$ may be determined by $L^2(\mu)$ projection of h onto $\mathrm{span}\{1, e^{-cq}, \ldots, e^{-cnq}\}$, where the "prior" μ is a finite measure on $[0,\infty)$. In practice, $a_{n,k}$ may be computed by weighted least squares regression of $h(q)$ on the regressors $\{q \mapsto e^{-ckq} : k = 0,\ldots,n\}$, with weights given by μ. Then [5] shows that (2.4) still holds, regardless of the choice of the prior μ, provided that $\mathrm{d}P/\mathrm{d}\mu$ exists in $L^2(\mu)$, where P denotes the $\mathbb{P}$-distribution of $[X]_T$.

For *hedging* purposes, the summation in the RHS of (2.4) provides a European-style payoff that, in conjunction with share trading, replicates the volatility payoff $h([X]_T)$ to arbitrary accuracy.

2.3 Pricing via variance distribution inference

Given the prices $\mathbf{c} \in \mathbb{R}^{N\times 1}$ of vanilla options at strikes $K_1, \ldots, K_N$, a scheme in [8] discretizes into $\{v_1, \ldots, v_J\}$ the possible values of $[X]_T$, and proposes to infer the discretized variance distribution $\mathbf{p} \in \mathbb{R}^{J\times 1}$ where $p_j := \mathbb{P}([X]_T = v_j)$, by solving approximately for $\mathbf{p}$ in

$$\mathbf{Bp} = \mathbf{c}, \tag{2.5}$$

where $\mathbf{B} \in \mathbb{R}^{N\times J}$ is given by $B_{nj} := C^{BS}(K_n, v_j)$, the Black-Scholes formula for strike K_n and squared unannualized volatility v_j. The approximate solution is chosen to minimize $||\mathbf{Bp} - \mathbf{c}||^2$ plus a convex penalty term. The contact paying $h([X]_T)$ is then priced as $\sum p_j h(v_j)$.

3 Pricing by use of variance or volatility swaps

With sufficient liquidity, variance and/or volatility swap quotes can be taken as inputs. For example, an approximation in [8] prices variance options by fitting a lognormal variance distribution to variance and volatility swaps of the same expiry. An approximation in [4] prices and hedges variance and volatility options by fitting a displaced lognormal, to variance and volatility swaps.

The variance curve models in [1] apply a different approach to using variance swaps; they take as inputs the variance swap quotes at multiple expiries, and they model the dynamics of the term structure of forward variance. Applications include pricing and hedging of realized variance options.

4 Pricing bounds by model-free use of Europeans

In this section, consider variance options on, more generally, any continuous share price Y.

Given European options of the same expiry T, there exist model-free sub/super-replication strategies, and hence lower/upper pricing bounds, for the variance options. Here *model-free* means that, aside from continuity and positivity, we make *no* assumptions on Y.

4.1 Subreplication and lower bounds

The following subreplication strategy is due to [7]; this exposition is based also on [3]. Let $\lambda : (0,\infty) \to \mathbb{R}$ be convex, let λ_y denote its left-hand derivative, and assume that its second derivative in the distributional sense has a density, denoted, λ_{yy}, which satisfies for all $y \in \mathbb{R}_+$

$$\lambda_{yy}(y) \leq 2/y^2. \tag{4.1}$$

Define for $y > 0$ and $v \geq 0$

$$BS(y, v; \lambda) := \begin{cases} \int_{-\infty}^{\infty} \lambda(ye^z) \, \frac{1}{\sqrt{2\pi v}} \exp\left[-\frac{(z+v/2)^2}{2v} \right] \mathrm{d}z & \text{if } v > 0 \\ \lambda(y) & \text{if } v = 0, \end{cases} \tag{4.2}$$

and let BS_y denote its y-derivative. Let $\tau_Q := \inf\{t \geq 0 : [X]_t \geq Q\}$. Then the following trading strategy subreplicates the variance call payoff: Hold statically a claim that pays at time T

$$\lambda(Y_T) - BS(Y_0, Q; \lambda), \tag{4.3}$$

and trade shares dynamically, holding at each time $t \in (0, T)$

$$\begin{aligned} -BS_y(Y_t, Q - [X]_t; \lambda) \qquad & \text{shares if } t \leq \tau_Q, \\ -\lambda_y(Y_t) \qquad & \text{shares if } t > \tau_Q, \end{aligned} \tag{4.4}$$

and a bond position that finances the shares and accumulates the trading gains or losses. Therefore the time-0 value of the contract paying (4.3) provides a lower bound on the variance call value.

The lower bound from (4.3) is optimized by λ consisting of $2/K^2 \mathrm{d}K$ OTM vanilla payoffs at all K where $I_0(K, T)$, the squared unannualized Black-Scholes implied volatility, exceeds Q:

$$\lambda(y) = \int_{\{K: I_0(K,T) > Q\}} \frac{2}{K^2} \mathrm{van}_K(y) \, \mathrm{d}K. \tag{4.5}$$

See [3] for generalization to forward-starting variance options.

4.2 Superreplication and upper bounds

The following superreplication strategy is due to [3]. Choose any $b_d \in (0, Y_0]$ and $b_u \in [Y_0, \infty)$. Let

$$BP(y, q;\ b_d, b_u) := \int_{-\infty-\alpha i}^{\infty-\alpha i} \frac{\sqrt{y/b_u}\sinh(\log(b_d/y)\sqrt{1/4-2iz}) - \sqrt{y/b_d}\sinh(\log(b_u/y)\sqrt{1/4-2iz})}{2\pi z^2 e^{i(Q-q)z}\sinh(\log(b_u/b_d)\sqrt{1/4-2iz})}\mathrm{d}z, \tag{4.6}$$

where $\alpha > 0$ is arbitrary. For $y > 0$ define

$$L(y;\ b_d, b_u) := \begin{cases} -2\log(y/b_u) + 2\frac{\log(b_u/b_d)}{b_u-b_d}(y-b_u) & \text{if } b_d \neq b_u \\ -2\log(y/Y_0) + 2y/Y_0 - 2 & \text{if } b_d = b_u = Y_0, \end{cases} \tag{4.7}$$

and

$$L^*(y;\ b_d, b_u) := \begin{cases} L(y;\ b_d, b_u) & \text{if } y \notin (b_d, b_u) \\ -BP(y, 0;\ b_d, b_u) & \text{if } y \in (b_d, b_u). \end{cases} \tag{4.8}$$

Let BP_y and L_y denote the y-derivatives, and let $\tau_b := \inf\{t \geq 0 :\ Y_t \notin (b_d, b_u)\}$.

Then the following strategy superreplicates the variance call payoff $([X]_T - Q)^+$. Hold statically a claim that pays at time T

$$L^*(Y_T;\ b_d, b_u) - L^*(Y_0;\ b_d, b_u), \tag{4.9}$$

and trade shares dynamically, holding at each time at each time $t \in (0, T)$

$$\begin{aligned} BP_y(Y_t, [X]_t - [X]_0;\ b_d, b_u) \qquad & \text{shares if } 0 \leq t \leq \tau_b \\ -L_y(Y_t;\ b_d, b_u) \qquad & \text{shares if } t > \tau_b, \end{aligned} \tag{4.10}$$

and a bond position that finances the shares and accumulates the trading gains or losses.

Therefore the time-0 value of the contract paying (4.9) provides an upper bound on the variance call value. Given T-expiry European options data, the upper bound from (4.9) may be optimized over all choices of (b_d, b_u).

4.3 Connection to the Skorohod problem

Whereas sections 4.1 and 4.2 presented explicit hedging strategies which imply pricing bounds, this section presents (a logarithmic version of) the result in [7], which showed that stopping-time analysis also implies pricing bounds.

Denote by ν the $\mathbb{P}$-distribution of Y_T, which is revealed by the prices of T-expiry options on Y.

Suppose $\tilde{Y}$ is a continuous $\mathcal{F}$-martingale with $\tilde{Y} \sim \nu$, and $[\tilde{X}]_T$ has finite expectation, where $\tilde{X} := \log \tilde{Y}$. Then Dambis-Dubins-Schwartz implies that $\tilde{Y}_t = G_{[\tilde{X}]_t}$, where G is a driftless unit-volatility geometric $\mathcal{G}$-Brownian motion (on an enlarged probability space if needed) with $G_0 = Y_0$, and $[\tilde{X}]_t$ are $\mathcal{G}$-stopping times, where $\mathcal{G}_s := \mathcal{F}_{\inf\{t:\ [\tilde{X}]_t > s\}}$. Thus $G_{[\tilde{X}]_T} \sim \nu$, hence $[\tilde{X}]_T$ solves a *Skorohod problem*: it is a finite-expectation stopping time that embeds the distribution ν in G. Conversely, if some finite-expectation τ embeds ν in a driftless unit-volatility geometric Brownian motion G, then $\tilde{Y}_t := G_{\tau\wedge(t/(T-t))}$ defines a continuous martingale with $\tilde{Y} \sim \nu$ and $[\log \tilde{Y}]_T = \tau$.

So distributions of stopping times solving the Skorohod problem are identical to distributions of realized variance consistent with the given price distribution ν. Skorohod solutions that have optimality properties, therefore, imply bounds on prices of variance/volatility options. In particular, *Root's* solution is known [9] to minimize the expectations of convex functions of the stopping time; the minimized expectation is, in that sense, a sharp lower bound on the price of a variance option. See also [EQF02/015].

5 Contract specifications in practice

In practice, the realized variance in the payoff specification is defined by replacing quadratic variation $[X]_T$ with an annualized discretization that monitors Y, typically daily, for N periods, resulting in a specification

$$\text{Annualization} \times \sum_{n=1}^{N} \Big(\log \frac{Y_n + D_n}{Y_{n-1}} \Big)^2 \tag{5.1}$$

where D_n denotes the discrete dividend payment, if any, of the nth period; no adjustment is made for any dividends deemed to be continuously paid, such as dividends on an index.

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