Corridor Variance Swap

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A corridor variance swap, with corridor C, on an underlying Y is a *weighted variance swap* on $X := \log Y$ (unless otherwise specified), with weight given by the corridor's indicator function:

$$w(y) := \mathbb{I}_{y \in C}. \tag{1}$$

For example, one may define an *up*-variance swap by taking $C = (H, \infty)$, and a *down*-variance swap by taking $C = (0, H)$, for some agreed H.

In practice, the corridor variance swap monitors Y discretely, typically daily, for some number of periods N, annualizes by a factor such as $252/N$, and multiplies by notional, for a total payoff

$$\text{Notional} \times \text{Annualization} \times \sum_{n=1}^{N} \mathbb{I}_{Y_n \in C} \left(\log \frac{Y_n}{Y_{n-1}} \right)^2. \tag{2}$$

If the contract makes dividend adjustments (as typical for contracts on single stocks but not on indices), then the term inside the parentheses becomes $\log((Y_n + D_n)/Y_{n-1})$, where D_n denotes the dividend payment, if any, of the nth period.

Corridor variance swaps accumulate only the variance that occurs while price is in the corridor. The buyer therefore pays less than the cost of a full variance swap. Among the possible motivations for a volatility investor to accept this trade-off, and to buy up (or down) variance are the following. First, the investor may be bullish (bearish) on Y. Second, the investor may have the view that the market's downward volatility skew is too steep (flat), making down-variance expensive (cheap) relative to up-variance. Third, the investor may be seeking to hedge a short volatility position that worsens as Y increases (decreases).

Model-free replication and valuation

The continuously-monitored corridor variance swap admits model-free replication by a static position in options and dynamic trading of shares, under conditions specified in the *weighted variance swap* article, which include all positive continuous semimartingale share prices Y under deterministic interest rates and proportional dividends.

Explicitly, one replicates using that article's (7), with payoff derived in [3]:

$$\lambda(y) = \int_{K\in C} \frac{2}{K^2}\mathrm{Van}(y,K)\mathrm{d}K, \tag{3}$$

where $\mathrm{Van}(y,K) := (K-y)^+\mathbb{I}_{K<\kappa} + (y-K)^+\mathbb{I}_{K>\kappa}$ for an arbitrary put/call separator κ.

Therefore, in the case that the interest rate equals the dividend yield (otherwise, see the weighted variance swap article), a replicating portfolio statically holds $2/K^2\mathrm{d}K$ out-of-the-money vanilla calls or puts at each strike K in the corridor C. The corridor variance swap model-independently has the same initial value as this portfolio of Europeans. Additionally, the replication strategy trades shares dynamically according to a "zero-vol" delta-hedge, meaning that its share holding equals the negative of what would be the European portfolio's delta under zero volatility.

For corridors of the type $C = (0,H)$ or $C = (H,\infty)$ where $H > 0$, taking $\kappa := H$ in (3) yields

$$\lambda(y) = (-2\log(y/H) + 2y/H - 2)\mathbb{I}_{y\in C}. \tag{4}$$

This λ, with H chosen arbitrarily, is also valid for the variance swap $C = (0,\infty)$.

Further properties

1. For a small interval $C = (a,b)$, the corridor variance swap approximates a contract on local time, in the following sense. Corridor variance satisfies

   $$V_T^{(a,b)} := \int_0^T \mathbb{I}_{X_t\in(\log a,\log b)}\mathrm{d}\langle X\rangle_t = \int_{\log a}^{\log b} L_T^x \mathrm{d}x,$$

   by the occupation time formula, where L_T^x denotes (an x-cadlag modification of) the *local time* of X. Therefore, at any point a,

   $$\frac{1}{\log b - \log a} V_T^{(a,b)} \longrightarrow L_T^a, \qquad \text{as } b \downarrow a.$$

2. Corridor variance can arise from imperfect replication of variance. The replicating portfolio for a standard variance swap holds options at all strikes $K \in (0,\infty)$. In practice, not all of those strikes actually trade. If we truncate the portfolio to hold only the strikes in some interval C, then the resulting value does not price a full variance swap but rather a C-corridor variance swap. (Moreover, in practice not even an interval of strikes actually trade, but rather a finite set, which can replicate instead a strike-to-strike notion of corridor variance, as shown in [1].)

3. In the case $C = (H,\infty)$ where $H > 0$, we rewrite (4) as

   $$\lambda(y) = \frac{2}{H}(y-H)^+ - 2(\log y - \log H)^+.$$

   Thus the replicating portfolio is long calls on Y_T and short calls on $\log Y_T$.

Let F_{X_T} be the characteristic function of $X_T = \log Y_T$. Then techniques in [4] and [5] price the calls on Y_T and $\log Y_T$ respectively. Specifically, assuming zero interest rates and dividends, we have the following semi-explicit formula for the corridor variance swap's fair strike:

$$\begin{aligned}\mathbb{E}\lambda(Y_T) - \lambda(Y_0) = \frac{2}{H\pi} \int_{0-\alpha i}^{\infty-\alpha i} \mathrm{Re}\Big(\frac{F_{X_T}(z-i)}{iz - z^2} e^{-iz\log H}\Big) \mathrm{d}z \\ + \frac{2}{\pi} \int_{0-\beta i}^{\infty-\beta i} \mathrm{Re}\Big(\frac{F_{X_T}(z)}{z^2} e^{-iz\log H}\Big) \mathrm{d}z - \lambda(Y_0),\end{aligned} \tag{5}$$

for arbitrary positive α, β such that $\alpha + 1,\ \beta < \sup\{p : \mathbb{E}Y_T^p < \infty\}$.

In the case $C = (0, \infty)$, equation (4) implies the fair strike formula

$$\mathbb{E}\lambda(Y_T) - \lambda(Y_0) = -2\mathbb{E}\log(Y_T/Y_0) = 2iF'_{X_T}(0) + 2\log Y_0. \tag{6}$$

In the case $C = (H_1, H_2)$ where $0 \leq H_1 < H_2$, subtract the formula for $C = (H_2, \infty)$ from the formula for $C = (H_1, \infty)$.

In the case of nonzero interest rates or dividends, add to (5) a correction involving payoffs at all expiries in $(0, T)$, as specified in *weighted variance swap* article's (7a); and in (6) replace the Y_0 by the forward price.

4. With discrete monitoring, the question arises, how to define up-variance and down-variance, and in particular how much variance to recognize, given a discrete move that takes Y across H. Definition (2) recognizes the full square of each move that ends in the corridor. Alternatively, the contract specifications in [2] treat the movements of Y across H by recognizing a fraction of the squared move. The fraction is defined in a way that admits approximate discrete hedging, in the sense that the time-discretized implementation of the continuous replication strategy has in each period a hedging error of only third-order in that period's return.

References

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[3] Peter Carr and Dilip Madan. Towards a theory of volatility trading. In R. Jarrow, editor, *Volatility*, pages 417–427. Risk Publications, 1998.

[4] Peter Carr and Dilip Madan. Option valuation using the fast Fourier transform. *Journal of Computational Finance*, 3:463–520, 1999.

[5] Roger Lee. Option pricing by transform methods: Extensions, unification, and error control. *Journal of Computational Finance*, 7(3):51–86, 2004.