Weighted Variance Swap

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February 17, 2009

Let the underlying process Y be a semimartingale taking values in an interval I. Let $\varphi : I \to \mathbb{R}$ be a difference of convex functions, and let $X := \varphi(Y)$. A typical application takes Y to be a positive price process and $\varphi(y) = \log y$ for $y \in I = (0, \infty)$.

Then [the floating leg of] a forward-starting *weighted variance swap* or *generalized variance swap* on $\varphi(Y)$ (shortened to "on Y" if the φ is understood), with *weight* process w_t, forward-start time θ, and expiry T, is defined to pay, at a fixed time $T_{\text{pay}} \geq T > \theta \geq 0$,

$$\int_\theta^T w_t \mathrm{d}[X]_t, \tag{1}$$

where $[\cdot]$ denotes quadratic variation. In the case that $\theta = 0$, the trade date, we have a spot-starting weighted variance swap. The basic cases of weights take the form $w_t = w(Y_t)$, for a measurable function $w : I \to [0, \infty)$, such as:

- The weight $w(y) = 1$ defines a *variance swap* [EQF07-024].
- The weight $w(y) = \mathbb{I}_{y \in C}$, the indicator function of some interval C, defines a *corridor variance swap* [EQF07-027] with corridor C. For example, a corridor of the form $C = (0, H)$ produces a *down* variance swap.
- The weight $w(y) = y/Y_0$ defines a *gamma swap* [EQF07-028].

Model-free replication and valuation

Assuming a deterministic interest rate r_t, let Z_t be the time-t price of a bond that pays 1 at time T_{pay}. Assume that Y is the continuous price process of a share that pays continuously a deterministic proportional dividend q_t. Let

$$Z_t = \exp\Big(- \int_t^{T_{\text{pay}}} r_u \mathrm{d}u \Big) \quad \text{and} \quad Q_t := \exp\Big(\int_0^t q_u \mathrm{d}u \Big), \tag{2}$$

so the share price with reinvested dividends is $Y_t Q_t$. Then the payoff

$$\int_\theta^T w(Y_t) \mathrm{d}[X]_t \tag{3}$$

admits a model-independent replication strategy, which holds European options statically, and trades the underlying shares dynamically. Indeed, let $\lambda : I \to \mathbb{R}$ be a difference of convex functions, let λ_y denote its left-hand derivative, and assume that its second derivative in the distributional sense has a signed density, denoted λ_{yy}, which satisfies for all $y \in I$

$$\lambda_{yy}(y) = 2\varphi_y^2(y)w(y), \tag{4}$$

where φ_y denotes the left-hand derivative of φ. Then

$$\begin{aligned}\int_\theta^T w(Y_t)\mathrm{d}[X]_t &= \lambda(Y_T) - \lambda(Y_\theta) - \int_\theta^T \lambda_y(Y_t)\mathrm{d}Y_t \quad (5)\\ &= \lambda(Y_T) - \lambda(Y_\theta) + \int_\theta^T (q_t - r_t)\lambda_y(Y_t)Y_t\mathrm{d}t \\ &\qquad - \int_\theta^T \lambda_y(Y_t)\frac{Z_t}{Q_t}\mathrm{d}(Y_tQ_t/Z_t), \quad (6)\end{aligned}$$

where (5) is by a proposition in [1] that slightly extends [2], and (6) is by Ito's rule. So the following self-financing strategy replicates (and hence prices) the payoff (3). Hold statically a claim that pays at time T_{pay}

$$\lambda(Y_T) - \lambda(Y_\theta) + \int_\theta^T (q_\tau - r_\tau)\lambda_y(Y_\tau)Y_\tau\mathrm{d}\tau, \tag{7a}$$

and trade shares dynamically, holding at each time $t \in (\theta, T)$

$$-\lambda_y(Y_t)Z_t \quad \text{shares}, \tag{7b}$$

and a bond position that finances the shares and accumulates the trading gains or losses. Hence the payoff (3) has time-0 value equal to that of the replicating claim (7a), which is synthesizable from Europeans with expiries in $[\theta, T]$. Indeed, for a put/call separator κ (such as $\kappa = Y_0$), if $\lambda(\kappa) = \lambda_y(\kappa) = 0$, then each λ claim decomposes into puts/calls at all strikes K, with quantities $2\varphi_y^2(K)w(K)\mathrm{d}K$:

$$\lambda(y) = \int_I 2\varphi_y^2(K)w(K)\mathrm{Van}(y, K)\mathrm{d}K, \tag{8}$$

where $\mathrm{Van}(y, K) := (K - y)^+\mathbb{I}_{K<\kappa} + (y - K)^+\mathbb{I}_{K>\kappa}$ denotes the vanilla put or call payoff. For put/call decompositions of general European payoffs, see [2].

Futures-dependent weights

In (3), the weight is a function of spot Y_t. The alternative payoff specification

$$\int_\theta^T w(Y_tQ_t/Z_t)\mathrm{d}[X]_t \tag{9}$$

makes w_t a function of the *futures* price (a constant times Y_tQ_t/Z_t).

In the case $\varphi = \log$, we have $[X] = [\log Y] = [\log(YQ/Z)]$, hence

$$\int_\theta^T w\big(Y_t Q_t/Z_t\big)\mathrm{d}[X]_t = \lambda(Y_T Q_T/Z_T) - \lambda(Y_\theta Q_\theta/Z_\theta) - \int_\theta^T \lambda_y(Y_t Q_t/Z_t)\mathrm{d}(Y_t Q_t/Z_t)$$

for λ satisfying (4). So the alternative payoff (9) admits replication as follows. Hold statically a claim that pays at time T_{pay}

$$\lambda(Y_T Q_T/Z_T) - \lambda(Y_\theta Q_\theta/Z_\theta), \tag{10a}$$

and trade shares dynamically, holding at each time $t \in (\theta, T)$

$$-\lambda_y(Y_t Q_t/Z_t)Q_t \quad \text{shares}, \tag{10b}$$

and a bond position that finances the shares and accumulates the trading gains or losses. Thus the payoff (9) has time-0 value equal to a claim on (10a).

In special cases (such as $w = 1$ or $r = q = 0$), the spot-dependent (3) and futures-dependent (9) weight specifications are equivalent. In general, the spot-dependent weighting is harder to replicate, as it requires a continuum of expiries in (7a), unlike (10a). The spot-dependent weighting is however the more common specification, and is assumed in remainder of this article.

Examples

Returning to the previously specified examples of weights $w(Y_t)$, we express the replication payoff λ in a compact formula, and also expanded in terms of vanilla payoffs according to (8). We take $\varphi(y) = \log y$ unless otherwise stated.

- Variance swap: Equation (4) has solution

$$\lambda(y) = -2\log(y/\kappa) + 2y/\kappa - 2 = \int_0^\infty \frac{2}{K^2}\mathrm{Van}(y, K)\mathrm{d}K.$$

- Arithmetic variance swap: For $\varphi(y) = y$, equation (4) has solution

$$\lambda(y) = (y - \kappa)^2 = \int_0^\infty 2\mathrm{Van}(y, K)\mathrm{d}K.$$

- Corridor variance swap: Equation (4) has solution

$$\lambda(y) = \int_{K \in C} \frac{2}{K^2}\mathrm{Van}(y, K)\mathrm{d}K.$$

- Gamma swap: Equation (4) has solution

$$\lambda(y) = \frac{2}{Y_0}\Big[y\log(y/\kappa) - y + \kappa\Big] = \int_0^\infty \frac{2}{Y_0 K}\mathrm{Van}(y, K)\mathrm{d}K.$$

In all cases, the strategy (7) replicates the desired contract. In the case of a variance swap, the strategy (10) also replicates it, because $w(Y) = 1 = w(YQ/Z)$.

Discrete dividends

Assume that at the fixed times t_m where $\theta = t_0 < t_1 < \cdots < t_M = T$, the share price jumps to $Y_{t_m} = Y_{t_m-} - \delta_m(Y_{t_m-})$, where each discrete dividend is given by a function δ_m of pre-jump price. In this case the dividend-adjusted weighted variance swap can be defined to pay at time T_{pay}

$$\sum_{m=1}^{M} \int_{t_{m-1}+}^{t_m-} w(Y_t)\mathrm{d}[X]_t. \tag{11}$$

If the function $y \mapsto y - \delta_m(y)$ has an inverse $f_m : I \to I$, and if Y is continuous on each $[t_{m-1}, t_m)$, then each term in (11) can be constructed via (7), together with the relation $\lambda(Y_{t_m-}) = \lambda(f_m(Y_{t_m}))$. Specifically, the mth term admits replication by holding statically a claim that pays at time T_{pay}

$$\lambda(f_m(Y_{t_m})) - \lambda(Y_{t_{m-1}}) + \int_{t_{m-1}}^{t_m} (q_\tau - r_\tau)\lambda_y(Y_\tau)Y_\tau \mathrm{d}\tau, \tag{12}$$

and holding dynamically $-\lambda_y(Y_t)Z_t$ shares at each time $t \in (t_{m-1}, t_m)$.

Contract specifications in practice

In practice, weighted variance swap transactions are forward-settled; no payment occurs at time 0, and at time T_{pay} the party long the swap receives the total payment

$$\text{Notional} \times \Big(\text{Floating} - \text{Fixed}\Big), \tag{13}$$

where "Fixed" (also known as the "strike"), expressed in units of annualized variance, is the price contracted at time 0 for time-T_{pay} delivery of "Floating," an annualized discretization of (11) which monitors Y, typically daily, for N periods. In the usual case of $\varphi = \log$, this results in a specification

$$\text{Floating} := \text{Annualization} \times \sum_{n=1}^{N} w(Y_n)\Big(\log \frac{Y_n + D_n}{Y_{n-1}}\Big)^2, \tag{14}$$

where D_n denotes the discrete dividend payment, if any, of the nth period. Both here and in the theoretical form (11), no adjustment is made for any dividends deemed to be continuous (for example, index variance contracts typically do not adjust for index dividends; see [3]).

In some contracts – for example, single-stock (down-)variance – the risk to the variance seller that Y crashes is limited by imposing a cap on the payoff. So

$$\text{Notional} \times \Big(\min(\text{Floating}, \text{Cap} \times \text{Fixed}) - \text{Fixed}\Big), \tag{15}$$

replaces (13), where "Cap" is an agreed constant, such as the square of 2.5.

References

[1] Peter Carr and Roger Lee. Hedging variance options on continuous semimartingales. Forthcoming in *Finance and Stochastics*, 2009.

[2] Peter Carr and Dilip Madan. Towards a theory of volatility trading. In R. Jarrow, editor, *Volatility*, pages 417–427. Risk Publications, 1998.

[3] Marcus Overhaus, Ana Bermúdez, Hans Buehler, Andrew Ferraris, Christopher Jordinson, and Aziz Lamnouar. *Equity Hybrid Derivatives.* John Wiley & Sons, 2007.

See also [EQF07-024], [EQF07-027], [EQF07-028], and the sources cited therein. I thank Peter Carr for valuable comments.