MATHEMATICAL COMPLEXITY OF SIMPLE ECONOMICS

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Abstract. Even simple, standard price adjustment models from economics – used to model the "invisible hand" story of Adam Smith – admit highly chaotic behavior. After relating these dynamical conclusions to complexity problems from numerical analysis and showing the mathematical reason why these results arise, it is suggested why similar counter-intuitive conclusions permeate the social sciences.

A lesson learned from modern dynamics is that natural systems can be surprisingly complex. No longer are we astonished to discover that systems from, say, biology (e.g., [GOI, Ma1, Ma2]) or the Newtonian n-body problem (e.g., [MM, Mo, Mk, SX, X]) admit all sorts of previously unexpected dynamical behavior. This seeming randomness, however, sharply contrasts with what we have been conditioned to expect from economics. On the evening news and talk shows, in the newspapers, and during political debate we hear about the powerful moderating force of the market which, if just left alone, would steadily drives prices toward an equilibrium with the desired balance between demand and supply. The way this story is invoked to influence government and even health policies highlights its important, critical role. But, is it true?

I have no idea whether Adam Smith's invisible hand holds for the "real world," but, then, no one else does either. This is because, even though this story is used to influence national policy, no mathematical theory exists to justify it. Quite to the contrary; what we do know indicates that even the simple models from introductory courses in economics can exhibit dynamical behavior far more complex than anything found in classical physics or biology. In fact, all kinds of complicated dynamics (e.g., involving topological entropy, strange attractors, and even conditions yet to be found) already arise in elementary models that only describe how people exchange goods (a pure exchange model).

Instead of being an anomaly, the mathematical source of this complexity is so common to the social sciences that I suspect it highlights a general problem plaguing these areas. If true, this assertion explains why it is difficult to achieve progress in the social sciences while underscoring the need for new mathematical tools. In

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this article I explain my suspicion by outlining what goes wrong with the price adjustment story. To do so, in Sect. 2, after quickly introducing the needed concepts (I recommend [V] for a complete, relaxed description), the price assertion is described in terms of the structure of vector fields on a sphere. Then, an explanation and extension of this counter-intuitive result are given.

2. The price model

In a $n \geq 2$ commodity world without production, agents can exchange goods according to (positive) prices. If p_j is the price per unit of the jth commodity, the cost of $x_j > 0$ units is $p_j x_j$. So, letting vector $\mathbf{p}$ represent the prices of all commodities, the cost of a *commodity bundle* $\mathbf{x} = (x_1, \ldots, x_n) \in R^n_+$ is computed by the inner product $(\mathbf{p}, \mathbf{x})$. In an exchange economy, what the kth agent can afford is based on what he can sell – his *initial endowment* $\mathbf{w}_k$ – which provides wealth $(\mathbf{p}, \mathbf{w}_k)$. Thus at prices $\mathbf{p}$, agent k can afford a commodity bundle $\mathbf{x}_k$ satisfying *the budget constraint* $(\mathbf{p}, \mathbf{x}_k) \leq (\mathbf{p}, \mathbf{w}_k)$, or any $\mathbf{x}_k$ in the *budget set*

$$\{\mathbf{x}_k \in R^n_+ \,|\, (\mathbf{p}, \mathbf{x}_k - \mathbf{w}_k) \leq 0\} \tag{2.1}$$

The boundary plane passing though $\mathbf{w}_k$ with the price vector $\mathbf{p}$ as a normal is the *budget plane.*

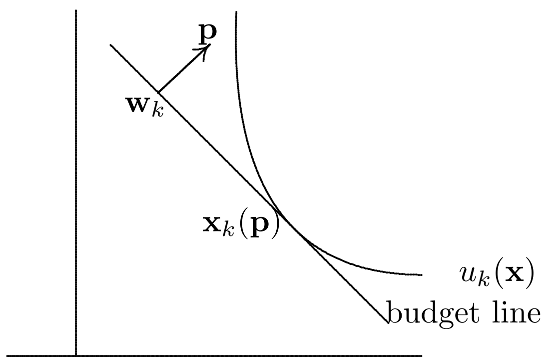

Fig. 1. The kth agent's demand function.

A person's choices at prices $\mathbf{p}$ are governed by personal preferences. As a natural ordering doesn't exist on R^n, $n \geq 2$, impose one by assuming each person's preferences are captured by a *utility function* $u_k : R^n_+ \to R$ where $u_k(\mathbf{y}) > u_k(\mathbf{x})$ iff the kth agent prefers bundle $\mathbf{y}$ to $\mathbf{x}$. To further simplify the mathematics, assume that individual preferences are strictly convex. This means that for any $\mathbf{x}$, those commodity bundles this person likes as much or better than $\mathbf{x}$, $\{\mathbf{y} \,|\, u_k(\mathbf{y}) \geq u_k(\mathbf{x})\}$, is a strictly convex set. Also, assume that all components of ∇u_k are positive. (Thus, all commodities are desired and an agent prefers more than less of each good.) With this idealized set-up, the kth agent's demand at price $\mathbf{p}$, $\mathbf{x}_k(\mathbf{p})$, can be determined by elementary Lagrange multiplier techniques; it is where a u_k level set is tangent to the budget plane. (See Fig. 1.) As this requires ∇u_k to be orthogonal to the budget plane at $\mathbf{x}_k(\mathbf{p})$, there is a positive scalar λ so that

$$\lambda \mathbf{p} = \nabla u_k(\mathbf{x}_k(\mathbf{p})). \tag{2.2}$$

The kth agent's *excess demand*, $\xi_k(\mathbf{p}) = \mathbf{x}_k(\mathbf{p}) - \mathbf{w}_k$, is the difference between what is demanded, $\mathbf{x}_k(\mathbf{p})$, and what is supplied, $\mathbf{w}_k$. This elementary derivation immediately leads to the classical properties of the *aggregate excess demand function,* $\xi(\mathbf{p}) = \sum_{k=1}^{n} \xi_k(\mathbf{p})$, called *Walras' laws.*

1. $\xi(\mathbf{p})$ is single-valued and smooth (because of u_k's convexity and smoothness),
2. $\xi(\mathbf{p})$ is homogeneous of degree zero (because each $\xi_k(\mathbf{p})$ is defined by the tangency of the utility function with the budget plane, and for any positive scalar μ, both $\mathbf{p}$ and $\mu\mathbf{p}$ define the same budget plane), and
3. $\xi(\mathbf{p})$ is orthogonal to $\mathbf{p}$ (because both $\mathbf{w}_k$ and $\mathbf{x}_k(\mathbf{p})$ are in the budget plane).

What else happens. As only elementary concepts are used, one might anticipate only well-behaved properties to emerge. But, as already promised, this is not true. To place this problem in a mathematically more convenient framework, notice that Prop. 2 allows us to scale the prices to norm 1; so, treat prices as points on the *price simplex* S^{n-1}_{+} – the intersection of the unit sphere S^{n-1} with the positive orthant R^n_+. On the price simplex, $\xi(\mathbf{p})$ is a smooth, tangent vector field (Prop. 1, 3).

Independent of how prices may change, it is reasonable to wonder whether equilibria for Adam Smith's story exist; namely, is there a price $\mathbf{p}^*$ whereby $\xi(\mathbf{p}^*) = \mathbf{0}$ so supply equals demand? To see why this is true, notice from the construction that choosing $\mathbf{p}$ nearly orthogonal to an axis forces the budget plane (the constraint for the Lagrange multiplier problem) to be nearly parallel to this axis. The strict convexity of preferences combined with the optimization procedure, then, forces a large excess demand for this good. This makes sense; the $\mathbf{p}$ choice significantly reduces the (relative) price of a desirable good, so its demand should become unbounded. Mathematically, this forces the vector field $\xi(\mathbf{p})$ to point toward the interior of the price simplex all along the boundary, so, from the Brouwer fixed point theorem (e.g., see [M]), $\xi(\mathbf{p})$ has a zero; thus, price equilibria exist. This description captures the essence of the important Arrow-Debreu construction [AD, AH, De2] establishing in quite general settings the existence of Adam Smith's equilibria.

Price equilibria exist, but do prices tend toward them? In differential form, the commonly told story about the price dynamic, where an increase in demand results in an increase in prices, is

$$\mathbf{p}' = \xi(\mathbf{p}), \tag{2.3}$$

with the discrete analogue

$$\mathbf{p}_{n+1} = \mathbf{p}_n + h\xi(\mathbf{p}_n) \tag{2.4}$$

for some positive constant h. In either setting, the resulting price dynamic is governed by the properties of $\xi(\mathbf{p})$. The natural question posed by Hugo Sonnenschein [So1, 2], then, is to determine all general properties beyond 1-3 that $\xi(\mathbf{p})$ must satisfy. For instance, *if* Adam Smith's invisible hand story holds, then at least one of the price equilibria must be a local attractor (where nearby prices converge to it). Or, if Eqs. 2.3,4 never are chaotic, or never have positive topological entropy, or never admit an attractor with a particular fractal dimension, or fail to satisfy the newest form of chaos yet to be discovered, then these conditions constitute still other properties enjoyed by excess demand functions.

To re-express Sonnenschein's question, let $\Xi(n)$ be the set of continuous tangent vector fields on S_+^{n-1}, $\mathcal{U}$ the set of continuous (smoothness is dropped as the tangency of a level set and the budget plane suffices), strictly convex utility functions, and R_+^n the space for initial endowments. With a agents, the construction of the aggregate excess demand function defines a mapping

$$\mathcal{F} : [\mathcal{U} \times R_+^n]^a \to \Xi(n) \tag{2.5}$$

Using this notation, we can interpret Sonnenschein's question as seeking a characterization of the $\mathcal{F}$ image set in $\Xi(n)$.

Sonnenschein provided an answer, Mantel [M] improved it, and Debreu [De1] proved the version of the SMD theorem which, in our notation, follows. In this theorem, $S_{+,\epsilon}^{n-1} = \{\mathbf{p} \in S_+^{n_1} \mid \text{each } p_j \geq \epsilon\}$ is a trimmed price simplex bounding prices away from zero, and $\Xi_\epsilon(n)$ is the set of continuous tangent vector fields on $S_{+,\epsilon}^{n_1}$.

SMD Theorem. *For $n \geq 2$ and $\epsilon > 0$, the price mapping*

$$\mathcal{F}_\epsilon : [\mathcal{U} \times R_+^n]^a \to \Xi_\epsilon(n) \tag{2.6}$$

is surjective iff $a \geq n$.

In other words, with at least as many agents as commodities, anything can happen! Whatever dynamic on $S_{+,\epsilon}^{n-1}$ is contemplated, no matter how complex, or how it may imitate a favored example from physics or the newest form of chaotic dynamics, the SMD theorem ensures there exist endowments and continuous, strictly convex preferences for the $a \geq n$ agents so that, at least on the trimmed price simplex, the aggregate excess demand function is the chosen vector field. It now is trivial to dismiss the Smith story simply by choosing a vector field of the kind illustrated in Fig. 2a with a lone, unstable equilibrium. While this economy admits an equilibrium, the prices move away from it. (For supporting preferences, see [Sc].) Of course, as we also could choose a vector field without a zero, the zero promised by the fixed point theorem must be hiding in the excised ϵ-region. On the other hand, should the vector field have the correct global index properties on the boundary of $S_{+,\epsilon}^{n-1}$, preferences can be found where $\xi(\mathbf{p})$ has no zeros in the excised region (which restores the original intent of the conclusion) [MC].

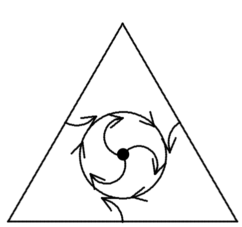

Fig. 2a. Scarf's example

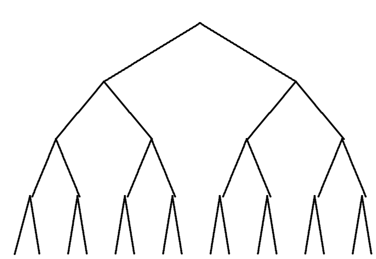

Fig. 2b. Coin flipping

3. Consequences of the SMD Theorem

What a mess! Although the SMD Theorem seriously erodes confidence in Smith's invisible hand, there are counterarguments. A natural approach is to dismiss SMD by speculating that only pathological preferences could define an aggregate excess demand function admitting chaos. To explore this hope, let me provide a quick, intuitive introduction for "chaos" by using the highly random event of continually flipping a coin. As either "Heads" or "Tails" can follow each event, the complexity is manifested by the tree diagram (Fig. 2b) where any downward branch lists all admissible outcomes. The dynamic governing the path next taken at each node is decided by chance, and the complexity is manifested by the rapid growth of the tree size generated by the two different choices that follow each event.

If the "chance moves" of a coin flip could be replaced with a deterministic dynamic $x_{n+1} = f(x_n)$, then the dynamics would admit an equally complicated structure – chaos. This can be accomplished if phase space can be divided into regions where the f image of each is the full space. (Examples are easy to construct; e.g., for $H = [0, a]$, $T = (a, 1]$, let the graph of f connect $(0, 0)$ with $(a, 1)$ with $(1, 0)$; this defines the well-analyzed tent map; e.g., [D, R].) The only difference between the tree diagrams for a flipping penny and this dynamic is that the path taken at each juncture is decided by deterministic dynamics; either choice is admissible and decided by the selection of the initial condition. So, whenever phase space can be divided into several regions where several admissible paths emerge from each node, the complexity defined by the deterministic dynamics can make the random tree of the flipping coin pale in comparison. (For an example with Newton's method for finding zeros of polynomials, see [S2]. In [S2], concepts from "chaos" are then modified to analyze "static" aggregation paradoxes from statistics, probability, and voting.) Each admissible branch on this tree for dynamics is called a *word*; the set of all words (i.e., all admissible branches) is the *dictionary*.

It now is clear how to construct a dynamic with as complicated a dictionary as desired. The main ingredient is for the map to be sufficiently expansive so that the f image over a specified region covers several other specified regions. As the SMD Theorem ensures that this "over reaction" effect occurs with price dynamics, it remains to understand whether it requires the preferences for agents to be so strained that economists could reject them – along with the troubling consequences of the SMD theorem – as being unrealistic. To analyze this inverse problem, notice that (with the exception of the scalar term λ) Eq. 2.2 resembles an inverse function relationship $x = g(g^{-1}(x))$ with its derivative condition $(g^{-1})' = 1/g'$. Thus, one might suspect from Eq. 2.2 that expansiveness of individual demands must, in some way, correspond to preferences where the level sets of u_k are fairly flat with small curvature. (While the precise conditions are more complicated [S6], this intuition is correct.) Now, by browsing through books and journals on mathematical economics, one discovers that this is a common choice for utility functions. Thus, the SMD theory cannot be rejected on these grounds.

Another way to try to save Smith's story is to accept the SMD theorem but wonder whether Eqs. 2.3, 4 are overly simplistic; maybe the market works in more complicated and mysterious ways. If so, then how should market mechanisms be modeled and what do they require to ensure that some price equilibria always will be reached? The long history of this theme (e.g., see [AH, AHu, H]) includes Smale's [Sm] "Globalized Newton Method (GNM)" (which can be viewed as using Milnor's [M] proof of the Brouwer fixed point theorem to extend [KLY].) Smale starts with a vector field, $f(\mathbf{x})$, on a k-dimensional simplex, so its normalized form,

$G(\mathbf{x}) = \frac{f(\mathbf{x})}{\|f(\mathbf{x})\|}$, maps the simplex to S^{k-1}. Because Sard's Theorem ensures that almost all values are regular, for almost all $\mathbf{c} \in S^{k-1}$ the closed set $G^{-1}(\mathbf{c})$ is the image of a finite union of circles and closed line intervals. As the endpoints of an image of a line interval must be in the boundary of the domain of G, they are either boundary points of the simplex, *or*, of more interest, zeros of f. Imposing appropriate boundary conditions on f (so that, at least generically, the direction defined by f at each point on the simplex boundary is unique) forces one endpoint to be a zero of f. Thus, by parameterizing the curve by $\alpha(t)$ and differentiating $G(\alpha(t)) = \mathbf{c}$, a zero is found by solving the differential equation $\alpha'(t) = \lambda D^{-1}_{\alpha(t)} f(f(\alpha(t))$ with an initial condition on the simplex boundary; this is a continuous version of Newton's method.

The GNM generated interest in economics with its guarantee of finding a zero for $\xi(\mathbf{p})$. But, quickly, it was dismissed as an explanation of price dynamics because there was no way to justify the market behaving in this contrived manner. Equally as important was the associated informational overload; not only does GNM feed on all the market information from $\xi(\mathbf{p})$, but also from its Jacobian $D_{\mathbf{p}}\xi$. Such information, requiring knowledge of, say, how the demand for steel varies with the price of bubble gum exceeds all bounds of decency and reasonableness making the GNM an unrealistic explanation of Smith's invisible hand.

If the GNM doesn't work, what does? Simon and I [SS] investigated this question by seeking the minimal conditions that would allow a market mechanism to work. Instead of a particular procedure, we assumed the general form

$$\mathbf{p}' = M(\xi(\mathbf{p}), D_{\mathbf{p}}\xi) \tag{3.1}$$

where M is piecewise smooth and where the dynamics stops iff $\xi(\mathbf{p}) = \mathbf{0}$; our goal was to find what kind of information does M need to ensure convergence to *some* price equilibrium.

Notice that Eq. 3.1 extends Eq. 2.3 by permitting a wide spectrum of possible choices ranging from where different commodities have different rates of price adjustment (by choosing $M = A(\xi(\mathbf{p}))$ where A is a positive diagonal matrix) to potentially very complex mechanisms. But even with all this flexibility, our result is discouraging for $n \geq 3$ commodities. Namely, should prices adjust as suspected – with some choice of M – then M needs most of the differential information required by GNM to always ensure convergence to at least one of the price equilibria. Some minimal informational savings can arise (i.e., some terms from the Jacobian can be dropped) when designing an M by exploiting the boundary properties of the vector field. (For instance, the integrated information requiring the excess demand to point inwards along the boundary of the price simplex exempts the two-good setting from this negative assertion; Eq. 2.3 does support the $n = 2$ supply and demand story. Here, the price simplex is the portion of the unit circle in the first quadrant. The boundary properties of $\xi(\mathbf{p})$ force an orbit of $\mathbf{p}' = \xi(\mathbf{p})$ starting near the boundary to move inward along the circle until it hits one of the promised price equilibria.) So, trying to preserve the Adam Smith story even in this general Eq. 3.1 framework carries the heavy cost of needing an unrealistic amount of information. (Among the extensions of [SS], I call attention to [J]. In [SS] we used the geometric theory of dynamical systems, the structure of $GL(n)$ and some singularity theory; [J] replaces the singularity theory with a topological argument.)

There is even more bad news; by being a differential equation Eq. 3.1 requires a continuum of market information. Of course, we would replace Eq. 3.1 with a discrete version, but could information be lost in the gaps? To investigate this question, in [S3] the discrete analogue

$$p_{n+1} = p_n + M(\xi(p_n), \xi'(p_n), \dots, \xi^{(s)}(p_n); \dots, \xi(p_{n-j}), \dots, \xi^{(s)}(p_{n-j})) \tag{3.2}$$

was analyzed to find the minimal conditions on M (the price mechanism), the number of derivatives s of the excess demand function, and the time lag j needed to ensure convergence of any exchange economy to some price equilibrium. (There are, of course, convergent mechanisms such as the bisection method, but we need Eq. 3.2 to investigate Smith's market pressure story and whether information is lost by using a discrete version of the differential equation.) Again, Eq. 3.2, which permits the simplistic approach of Eq. 2.4 to be replaced with highly complex, involved mechanisms, appears to subsume most, if not all, proposed price theories. Nevertheless, the theorem asserts that with $n \geq 2$ commodities, no mechanism can always promise convergence to a price equilibria. Instead, because of the admissible chaotic behavior of Eq. 3.2 inherited from the SMD Theorem, for any choice of M, s, and j, there exists an open set of aggregate excess demand functions (in any reasonable topology on function space) and an open set of initial conditions where convergence never occurs. (For an interesting variation of this impossibility assertion, see [BK].) Incidentally, the same assertion holds for numerical methods used to find the real zeros of real polynomials [S4].

A partial positive conclusion finally was found in [SW]. Stated in words, if demands are driven by preferences, why should the economics of prices be the same in Rio as in Chicago, in Moscow as in Stockholm, or in Zurich as in Paris? Maybe the economists' long time goal of an universal price mechanism is an impossible dream; instead, maybe different locales require different mechanisms. To express this mathematically, say that a given mechanism M *covers a set of economies* (i.e, a set of individual preferences and initial endowments for the agents) *if* $p_{n+1} = p_n + M(\xi(p_n))$ *converges to at least one equilibrium should the prices start sufficiently close to it.* So, mimicking the the reason it is impossible to represent the sphere S^2 with a single chart, maybe the topology of price adjustments requires more than one mechanism to cover the set of all economies.

While Williams and I found (in a more general setting) that this is true, we also found that the space of economies is σ-compact with this topology where the obstacles preventing compactness are singularities. So, for any $\epsilon > 0$, if one is willing to exclude a set of economies of (an appropriate) measure less than ϵ (which eliminates a region around singularities), the remaining set of economies (i.e., the remaining choices of initial endowments and preferences) are covered by a finite number of price adjustment procedures. A successful mechanism exists for each economy, but we don't know which one. To relate this assertion to actual practice, notice that the purpose of "market regulations" is to change the price dynamic. So these results imply that while an unregulated free market might not work as widely advertised, if *correct* regulations are imposed, the market now might behave as desired. This conclusion probably would not be to Smith's liking, but it finally is a positive assertion and we might not be able to do much better.

4. More difficulties

Dampening this partial success story is the realization that the situation is much more complex. Start with the fact that by measuring market reactions, the aggregate excess demand function is an important tool used in all sorts of ways. For instance, in spite of the SMD Theorem, it is reasonable to use a specified $\xi(\mathbf{p})$ to extract valuable information about the economy. Perhaps we can learn what to expect should a new commodity be added, or another suppressed. Similarly, an area called *consumer surplus* computes the excess demand functions for each commodity and then pieces this information together to obtain conclusions about the full excess demand. The area of macroeconomics, with its concern about measuring $\xi(\mathbf{p})$, quickly encounters realism where any attempt to compute $\xi(\mathbf{p})$ for all ten million commodities would generate Gee-Whiz comments "If a thousand modern computers started during the 'Big Bang,' ..." Consequently they use statistical measures of the excess demand for certain commodities to make inferences about the general situation.

All of these topics involve the tacit assumption that, in some way, the excess demand function for different sets of economies are related. But, are they? Must a well behaved economy of ten goods remain well behaved if one good is taken off the market, or could it become highly chaotic? To explore the reality of this assumption, it is natural to mimic Sonnenschein's question by seeking all possible relationships admitted by the aggregate excess demand functions with changes in the set of commodities. To describe this issue with the tree description used to introduce chaos, label the $2^n-(n+1)$ subsets of two or more commodities in some manner as $C_1, C_2, \dots, C_{2^n-(n+1)}$ where $|C_j|$ is the cardinality of C_j. As above, the aggregate excess demand function for the C_j commodities is a tangent vector field on $S_+^{|C_j|-1}$ and the set of all continuous tangent vector fields is denoted by $\Xi(C_j)$. The tree diagram starts with the uncountable number of choices from $\Xi(C_1)$ emerging from the C_1 node. Attached to each choice are the $\Xi(C_2)$ vector fields representing all of the C_2 choices. This continues until at the last node we have the $C_{2^n-(n+1)}$ vector fields. Now, if the implicit assumption that the behavior of the aggregate excess demand function for certain choices of commodities affects what happens with others is true, then certain branches can be pruned off of this chaotic tree of possibilities.

The answer again involves a trimmed price simplex which ignores (relative) prices lower than a specified ϵ value, and $\Xi_\epsilon(C_j)$ which denotes the continuous tangent vector fields on this trimmed simplex. (When the goods from C_j are traded, each agent holds fixed her holdings of all other commodities.)

Theorem ([S5]). *Let $\epsilon > 0$ be given. For $n \geq 2$ commodities, the mapping*

$$\mathcal{F}_\epsilon : [\mathcal{U} \times R_+^n]^a \to \prod_{j=1}^{2^n-(n+1)} \Xi_\epsilon(C_j) \tag{4.2}$$

is surjective iff $a \geq n$.

In other words, the "excess demand" tree description is full and chaotic; anything and everything can happen. This permits us to design all sorts of disturbing scenarios such as where with four goods the aggregate excess demand function carefully adheres to Smith's story with a single globally attracting price equilibrium. Then, withholding commodity c_j from the market creates a chaotic three-commodity vector field with an attractor of, say, fractal dimension $1 + \sqrt{\frac{j}{5}}$, $j = 1, \dots, 4$. The reader can choose what happens for each of the six two-good cases. According to the theorem, this scenario is supported by one of those deceptively innocent appearing four-agent examples where each agent is assigned a preference for goods (of the well behaved type described earlier) and an initial endowment. The market pressures, as found by simple Lagrange multiplier arguments, do the rest.

The theorem can be extended to address other concerns from economics. Namely, for any positive integer K, choose K continuous, tangent vector fields for each C_j set. Then, there exists an n-agent example where each agent is assigned a fixed preference relationship for the n commodities and K different initial endowments. Computing the aggregate excess demand with these preferences and the ith assignment of the initial endowments, we obtain (on the trimmed simplex) for each C_j its ith assigned vector field. To illustrate with $K = 2$ and only the full set of three commodities, the three agents' preferences could define a well behaved aggregate excess demand function that would delight Adam Smith should they use one set of initial endowments, but, using different endowments with the same preferences, any imaginable (two-dimensional) form of chaos can break out!

In other words, the SMD Theorem describes what happens with the single set of all commodities and a single assignment of initial endowments; the above result extends this disturbing conclusion to all sets of commodities and it shows that the conclusion can vary significantly with changes in endowments. In particular, this more general conclusion not only causes worry about the invisible hand story, but it forces us to question those tacit assumptions – assumptions basic to several tools from economics – about how the aggregate excess demand function for one commodity set relates to that of others. One might argue (and this is a common reaction during a colloquium lecture – particularly in a department of economics) that there may exist conditions imposing strong relationships. Yes, but it is obvious from the theorem that such constraints cannot be based upon the aggregate excess demand function (as is a common practice); instead they appear to require imposing unrealistically harsh global restrictions on the agents' preferences – restrictions similar to those shown in [CM] to be needed to justify the consumer surplus approach.

5. Idea behind the proofs

What is going on? The derivation of the aggregate excess demand function and Walras' laws is sufficiently elementary to be taught in a first course on vector calculus. So, we must wonder what a nice, simple model is doing in a complex place like this. Actually, the source of the difficulty – which is common across the social sciences – is that the social sciences are based on aggregation procedures. But, even simple aggregation methods, from probability, statistics, and even voting, admit surprisingly complex paradoxes. (For a description of some of them and why they occur, see [S1, 2].) One way to envision the aggregation difficulties is to recognize that even a simple mapping can admit a complex image should its domain have a larger dimension than its image space. This is an element of the proof of the last theorem, and it explains the hidden complexity of the social sciences. Namely, the complexity of the social sciences derives from the unlimited variety in individual preferences; preferences that define a sufficiently large dimensional domain that, when aggregated, can generate all imaginable forms of pathological behavior.

To prove the theorem, after selecting a tangent vector field for each $S_+^{|C_j|-1}$, we need to construct a continuous foliation for each agent so that the leaves (indifference sets of preferences; i.e., the level sets of the u_k functions) have the desired convexity properties and the excess demand function they define (from the Lagrange multiplier argument) agrees with the specified vector field on the trimmed simplex. While the construction is technical and difficult in places, intuition can be provided why the result holds and why we need as many agents as commodities. Start with a two-commodity utility function $u = xy$ where the level sets are hyperbolas. By experimenting with the Lagrange multiplier argument and different initial endowments, it becomes clear that the demand based on an initial endowment with a relatively small amount of one commodity favors this scarce commodity – we want what we don't have. Thus, with two agents with initial endowments emphasizing a different commodity, their excess demands point in opposing directions with zero in the convex hull. So, just by changing the magnitude of each individual excess demand at a price $\mathbf{p}$ (by flattening the level sets), the sum can realize a specified value of $\xi(\mathbf{p})$.

More generally, with any number of commodities, the excess demand associated with an initial endowment lacking in a particular commodity tends to favor that good. Therefore, with n commodities, choosing the jth agent to be shy in the jth good creates an excess demand function pointing in a particular direction; with $a \geq n$ agents, the convex hull defined by these vectors (in the tangent space at $\mathbf{p}$ of the price simplex) includes the origin as an interior point. Clearly, this is false for $a < n$. By varying the lengths of these vectors (i.e., by varying the curvature of the leaves of the foliation), their sum can be whatever we desire. In this way a continuous foliation is defined to do as advertised.

To connect the foliations defined for the various sets of commodities, notice that if a commodity is withheld from the market, the relevant portion of R^n_+ is an affine plane passing through $\mathbf{w}^k$ orthogonal to the axis of the missing commodity. As such, the budget plane passes through a portion of each level set much different from that used to construct the leaves for the larger sets of commodities. Now, the ϵ restriction forces a spacing among the foliations constructed for the different subsets of commodities. Part of the proof shows how to exploit this gap to connect the leaves from the different foliations to create a single foliation (while preserving the convexity properties, etc.). It follows immediately from this construction that any assertion trying to relate the aggregate excess demand functions from different subsets of commodities must impose strict restrictions on preferences. Therefore, constraints based only on the structure of the excess demand (as is typical) are doomed for failure. Also, since individual preferences drive the social sciences, this situation, with the accompanying troublesome complexity, undoubtedly extends to most other areas.

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