M. GROMOV

Entropy, homology and semialgebraic geometry

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ENTROPY, HOMOLOGY AND SEMIALGEBRAIC GEOMETRY
[after Y. Yomdin]

by M. GROMOV

1. COMPUTATIONAL DEFINITION OF TOPOLOGICAL ENTROPY

1.1. The *entropy* of a partition Π of a set X into N subset is defined by

$$\text{ent}\,\Pi = \log N \quad .$$

The *intersection* of two partition say $\Pi_1 \cap \Pi_2$, is the partition of X into the pairwise intersections of the elements of Π_1 and Π_2.

For a map $g : Y \to X$ one obviously defines the pull-back partition of Y denoted Π_g for every partition Π of X. If f is a self mapping $X \to X$ one consideres the pull-backs of Π under the iterates $f^1 = f$, $f^2 = f \circ f \ldots f^i = f \circ f^{i-1}$ and set

$$\Pi^i = \Pi \cap \Pi_f \ \ldots \cap \Pi_{f^i}$$

and

$$\text{ent}(\Pi; f, i) = i^{-1}\,\text{ent}\ \Pi^i \ .$$

Similarly, if Y is mapped into X by g one defines

$$\text{ent}(\Pi|Y; f, i) = i^{-1}\,\text{ent}(\Pi^i)_g \quad .$$

1.2. Let X be a cubical polyhedron, that is a topological space divided into cubes $\square$, such that every two cubes meet at a common face. Denote by Π the partition of X into the open (i.e. taken without boundary but not necessarily open as subsets in X) cubes of the polyhedron X and let $\Pi(j)$ be the refinement of Π obtained by dividing every $\square$ into $j^{\dim\square}$ equal subcubes. Now define the *topological entropy* $\text{ent}\, f$ of a map $f : X \to X$ as the lower bound of the numbers $h \geq 0$ with the following property :

(P) *There exists an arbitrarily large integer* $k \geq 0$ (*depending on* h) *such that*

$$\limsup_{i\to\infty} \text{ent}(\Pi(j)\ ;\ f^k, i) \leq hk$$

for all $j = 1,2,\ldots$.

In the same way one defines $\operatorname{ent} f|Y$ for every space Y mapped into X.

This definition is justified by the following easy theorem.

1.3. Topological invariance of the entropy. *If* X *is compact and* f *is continuous then* $\operatorname{ent} f$ *does not depend on a choice of the (cubical) polyhedral structure on* X. *The same applies to* $\operatorname{ent} f|Y$ *for compact spaces* Y *continuously mapped into* X. *Moreover, if* X *is finite dimensional and* $Y \subset X$ *is a compact subset invariant under* f *then* $\operatorname{ent} f|Y$ *only depends on* Y *and* $f : Y \to Y$ *(but not an embedding* $Y \to X$ *), provided the map* f *is continuous on* Y.

1.4. *Remark.* Consider *the standard partition* Π_{st} of $\mathbb{R}^n$ into unit cubes which are the faces of the integer translates of the cube $\{0 \le x_i \le 1 \ , \ i = 1 \dots n\} \subset \mathbb{R}^n$. The entropy defined with this Π_{st} is not topologically invariant over all $\mathbb{R}^n$. Yet it is invariant on every compact subset Y, such that f is continuous on Y and $f(Y) \subset Y$. Thus one obtains an invariant entropy for a continuous selfmaps of an arbitrary finite dimensional compact space Y, since Y embeds into some $\mathbb{R}^n$.

1.4. *Examples.* (A) Take a linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ and define *the spectral radius*

$$\operatorname{Rad} f = \lim \|f^i\|^{1/i}$$

for

$$\|f\| = \sup_{\|x\|=1} \|f(x)\| \quad .$$

Let $\Lambda_* f = \Lambda_\circ f \oplus \Lambda_1 f \oplus \dots \oplus \Lambda_n f$ be the full exterior power of f. Then by an easy argument, the entropy (for the standard cubical partition of $\mathbb{R}^n$) satisfies,

$$\operatorname{ent} f|Y = \log \operatorname{Rad} \Lambda_* f$$

for every non-empty open bounded subset Y in $\mathbb{R}^n$.

(B) Let f be an endomorphism of the torus $T^n = \mathbb{R}/\mathbb{Z}^n$. It is easy to see that

$$\operatorname{ent} f = \operatorname{ent} \tilde{f}|Y$$

for the covering linear map $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ and for every non-empty bounded open subset $Y \subset \mathbb{R}^n$. It follows with (A) that

$$\operatorname{ent} f = \log \operatorname{Rad} f_*$$

for the *induced endomorphism* f_* on the real homology $H_*(T^n)$.

(C) Every *holomorphic* map $f : \mathbb{C}P^n \to \mathbb{C}P^n$ has

$$\operatorname{ent} f = \log \operatorname{Rad} f_* \ . \tag{$*$}$$

Furthermore, $\operatorname{ent} f|Y = \operatorname{ent} f$ for every subset $Y \subset \mathbb{C}P^n$ whose complement is nondense and invariant under f. For example, if f on $\mathbb{C}P^1$ is given by a polynomial f_o on $\mathbb{C}^1 \subset \mathbb{C}P^1$ of degree $d > 0$, then $\text{ent } f|Y = \log d$ for $Y = \{|z| \leq r\} \subset \mathbb{C}$, provided $|f(z)| \geq r$ for $z \geq r$.

Notice that $\text{Rad } f_*$ equals the topological degree $\deg f$ for every continuous selfmap f of $\mathbb{C}P^n$ with $\deg f > 0$.

The proof of (*) consists of showing that

(C1) $$\text{ent } f \geq \log \deg f$$

and

(C2) $$\text{ent } f \leq \log^+ \deg f ,$$

where $\log^+ t = \max (0, \log t)$, which takes care of $\deg = 0$.

The first inequality is an immediate corollary of the following theorem by Misiurewicz and Przyticki (see $[\text{M-P}]_1$).

1.5. Theorem. *Let* f *be a* C^1*-smooth self-mapping of a compact manifold* X, *such that the pull back* $f^{-1}(x)$ *contains at least* d *point for all* x *in a subset of full measure in* X. *Then* $\text{ent } f \geq \log d$.

The second inequality (C2) follows from the (obvious) bound

$$\text{Vol } \Gamma_{f^i} \leq \text{const } d^i$$

for the 2n-dimensional volumes of the graphs $\Gamma_{f^i} \subset \mathbb{C}P^n \times \mathbb{C}P^n$ of the iterates of f. (See 2.4.)

1.6. Elementary properties of the entropy.

The following list of facts (whose proofs are straightforward) gives some idea on the dynamical significance of the entropy.

(i) For any two subsets in X,

$$\text{ent } f|Y_1 \cup Y_2 = \max_{i=1,2} \text{ ent } f|Y_i .$$

(ii) If $Y_1 \subset Y_2$ then $\text{ent } f|Y_1 \leq \text{ent } f|Y_2$.

(iii) Take two continuous selfmappings of compact spaces, say $f_i : X_i \to X_i$ for $i = 1,2$ and let $F : X_1 \to X_2$ be a continuous map commuting with f_i. If F is onto, then $\text{ent } f_1 \geq \text{ent } f_2$. If F is finite-to-one then, $\text{ent } f_1 \leq \text{ent } f_2$.

(iv) Suppose a continuous map $f : X \to X$ fixes a closed subset $X_o \subset X$ and *wanders* on the complement $\Omega = X \setminus X_o$. That is each point $x \in \Omega$ admits a neighborhood U such that $f^i(U)$ does not meet U for all sufficiently large i. Then $\text{ent } f = 0$, provided X is compact.

Examples. (a) Let f be a linear selfmapping of $\mathbb{R}^2$ with two real eigenvalues

$\neq \pm 1$. Such an f wonders outside the origin but $\operatorname{ent} f|Y$ may be positive on bounded subsets Y in $\mathbb{R}^2$ (see 1.4.A.). Next we extend f to a projective self-mapping $\bar{f}$ of the projective plane $P^2 \supset \mathbb{R}^2$. This $\bar{f}$ fixes, besides the origin in $\mathbb{R}^2$, two points on the projective line $P^1 = P^2 \setminus \mathbb{R}^2$ corresponding to the two eigenspaces (if the eigenvalues are equal $\bar{f}$ fixes P^1) and again $\bar{f}$ wanders outside the fixed point set. Since P^2 is compact, $\operatorname{ent} \bar{f} = 0$ by (iv) (compare (C.2) above. (Notice that $\operatorname{ent} f|Y \neq \operatorname{ent} \bar{f}|Y$ for $Y \subset \mathbb{R}^2 \subset P^2$ as the entropy in $\mathbb{R}^2$ defined with the standard cubical partition of $\mathbb{R}^2$ does depend on the partition and is not topologically invariant).

(b) Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given in the polar coordinates by $f : (\rho,\theta) \to (2\rho, d\theta)$ for some $\lambda > 1$ and an integer d . This f obviously extends to a continuous selfmap $\bar{f}$ of the one-point compactification of $\mathbb{R}^2$, that is $S^2 \supset \mathbb{R}^2$. The map $\bar{f}$ wanders outside the two (obvious) fixed points. Thus $\operatorname{ent} \bar{f} = 0$ and $\bar{f}$ violates the inequality $\operatorname{ent} \geq \log|\deg|$ for $|d| \geq 2$ (here $\deg \bar{f} = d$) as well as Theorem 1.5. This is due to the non-smoothness of f at the origin $0 \in \mathbb{R}^2$.

2. ENTROPY AND THE VOLUME GROWTH

2.1. Let X be a smooth Riemannian manifold (e.g. a submanifold in $\mathbb{R}^n$) and $f : X \to X$ a C^1-smooth maps. Take an ℓ-dimensional submanifold $Y \subset X$ and define

$$\operatorname{logvol}(f|Y) = \limsup_{i\to\infty} i^{-1} \log \operatorname{Vol}(\Gamma_{f^i}|Y)$$

where $\Gamma_{f^i}|Y \subset Y \times X$ stands for the graph of the i-th iterate of f on Y and Vol denotes the ℓ-dimensional Riemannian volume.

Notice that logvol can be bounded by the norm of the differential $Df : T(X) \to T(X)$,

$$\operatorname{logvol}(f|Y) \leq \log^+ ||Df||^{\ell} ,$$

where
$$||Df|| \overset{\text{def}}{=} \sup_x ||Df \mid T_x(X)|| .$$

The same estimate (obviously) holds with Rad Df instead of $||Df||$, where

$$\operatorname{Rad} Df \overset{\text{def}}{=} \limsup_{i\to\infty} ||Df^i||^{1/i} .$$

Observe that $\operatorname{Rad} Df \leq ||Df||$ and that Rad Df (unlike $||Df||$) does not depend on a choice of the Riemannian metric on X , provided X is compact.

2.2. YOMDIN THEOREM. *Let f be a C^r-smooth self-map of a compact C^∞-manifold X and let $Y \subset X$ be a compact C^r-submanifold. Then*

$$\operatorname{logvol}(f|Y) \leq \operatorname{ent}(f|Y) + \log^+(\operatorname{Rad} Df)^{\ell/r} . \qquad (*)$$

In particular, if f *and* Y *are* C^∞ *, then*

$$\operatorname{logvol}(f|Y) \leq \operatorname{ent}(f|Y) \leq \operatorname{ent} f . \qquad (**)$$

2.3. COROLLAIRE. *(Solution of Shub entropy conjecture for* C^∞*-maps). If* f *is* C^∞*-smooth then*

$$\operatorname{ent} f \geq \log \operatorname{Rad} f_* \qquad (***)$$

for the spectral radius Rad f_* *of the induced endomorphism on the real homology,* $f_* : H_*(X) \to H_*(X)$.

PROOF. Consider pairs of closed forms w_1 and w_2 on $X \times X$ with $\deg w_1 + \deg w_2 = \dim X$ and observe that

$$\operatorname{Rad} f_* = \operatorname{Rad} f^* = \sup_{w_1, w_2} \limsup_{i\to\infty} \left| \int_{\Gamma_{f^i}} w_1 \vee w_2 \right|^{1/i} \leq \limsup_{i\to\infty} (\operatorname{vol} \Gamma_{f^i})^{1/i} .$$

Remark. The spectral radius of f_* on H_ℓ is obviously bounded by the volume growth of the ℓ-simplices of fixed triangulation of V under the iterates of f .

2.3.A. *Example*. If f wanders outside the fixed point set of f (see 1.6. (iv)) then every eigenvalue λ of f_* on $H_*(x)$ satisfies $|\lambda| \leq 1$.

2.4. An upper bound for the entropy

Several months prior to Yomdin's result, Sheldon Newhouse [N] found the following converse to (**) for C^2-selfmaps of compact manifolds,

$$\operatorname{ent} f \leq \sup_Y \operatorname{logvol}(f\ Y) \qquad (****)$$

over all compact C^∞-submanifolds $Y \subset X$. A similar inequality for diffeomorphisms was proven earlier by Felix Przyticki [P] .

2.5. Semicontinuity of the entropy

Using (****) and his main lemma (see 3.4) Yomdin shows that

$$\limsup_{\tau\to 0} \operatorname{ent} f_\tau \leq \operatorname{ent} f_0$$

for every C^∞-continuous in $\tau \in [0,1]$ family of C^∞-maps $f_\tau : X \to X$ of a compact manifold X .

Example of non-continuous entropy

Map the unit disk in $\mathbb{C}$ into itself by $f_\tau : z \to (1-\tau)z^2$ for $\tau \in [0,1]$. Then $\operatorname{ent} f_0 = \log 2$ (see 1.4.C.) and $\operatorname{ent} f_\tau = 0$ for $\tau > 0$ as f_τ wanders outside the center of the disk for $\tau > 0$.

2.6. Yomdin's inequality (*) is sharp. To see this, let $Y \subset \mathbb{R}^2$ be the graph of the function $Y = x^{r+\varepsilon} \sin x^{-1}$ for $x \in [0,1]$ which is C^r-smooth for all r and $\varepsilon > 0$. Take the projective map f on $P^2 \supset \mathbb{R}^2$ given by the linear map $(x,y) \to (\frac{1}{2}x, 2y)$ of $\mathbb{R}^2$. Then the length of $f^i(Y)$ is about $2^{\frac{i}{r+\varepsilon}} = (\mathrm{Rad}\, Df)^{\frac{i}{r+\varepsilon}}$, while $\mathrm{ent}\, f = 0$. This makes $(\star)$ sharp for $\varepsilon \to 0$. If one insists on a C^∞- smooth Y and a C^r-smooth f then one just appropriately changes the smooth structure on P^2.

2.7. Several historical remarks

The relation between entropy and topology was discovered by Dinaburg [D] who observed that the time one map f^1 of the geodedic flow of a compact Riemannian manifold V has $\mathrm{ent}\, f^1 > 0$ if the fundamental group $\pi_1(V)$ has exponential growth. This is seen by looking at the universal covering of V and applying the following simple fact (compare 1.4.B) to the associated covering of the tangent bundle of V,

(A) *Let* $\tilde{X} \to X$ *be a Galois covering of a finite (cubical) complex* X *and let a continuous map* $f : X \to X$ *lift to a continuous map* $\tilde{f} : \tilde{X} \to \tilde{X}$. *If a compact subset* $Y \subset \tilde{X}$ *projects onto* X, *then*

$$\mathrm{ent}\, \tilde{f}|Y = \mathrm{ent}\, f ,$$

where one computes $\mathrm{ent}\, \tilde{f}$ *for the induced cubical structure on* $\tilde{X}$.

Notice that Yomdin's inequality (**) also yields $\mathrm{ent}\, f^1 > 0$ for C^∞-*smooth* V (Dinaburg's proof only needs the continuity of the geodesic flow). In fact, the inequality $\mathrm{ent}\, f^1 > 0$ follows from (**) for all C^∞-smooth V, where every two generic point, are joined by at least C^λ geodesic segments of length $\leq \lambda$ for all $\lambda \geq 1$ and some $C > 1$. This lower bound on the number of geodesic segments is satisfied for example, by those *simply connected* manifolds V for which the Betti numbers b_i of the loop space of V grow exponentially in $i = 1,2,\ldots$ (see [G]).

(B) Manning [Ma] proved that the spectral radius of $f_* : H_1(X) \to H_1(X)$ provides the (lower) bound

$$\mathrm{ent}\, f \geq \log \mathrm{Rad}\, f_*|H_1(X)$$

for every *continuous* map f of a compact polyhedron X (to see this apply (A) to the maximal Abelian cover $X \to X$) and Misiurewicz and Przyticki [M-P$_2$] sharpened this inequality for X homotopy equivalent to the n-torus,

$$\mathrm{ent}\, f \geq \log \mathrm{Rad}\, f_* = \log \mathrm{Rad}\, \Lambda_* f_*|H_1(X) .$$

(C) Shub conjectured that ent $f \geq \log \operatorname{Rad} f_*$ is satisfied by C^1-maps on all manifolds (see (b) in 1.6. for a C°-counterexample). Now, this conjecture is settled (besides tori) for C^1-maps of the spheres S^n (by 1.5.) and for C^∞-maps on all X by Yomdin's (***) .

3. REDUCTION OF YOMDIN THEOREM TO AN ALGEBRAIC LEMMA

3.1. C^r-size of a submanifold

Fix an integer $\ell = 1,2,..$ and define the C^r-*size* of a subset $Y \subset \mathbb{R}^n$ as the lower bound of the numbers $s \geq 0$ for which there exists a C^r-map of the unit ℓ-cube into $\mathbb{R}^n$, say $h : [0,1]^\ell \to \mathbb{R}^n$, whose image contains Y and such that $\|D_r h\| \leq s$. Here $D_r h$ is the vector assembled of (the components of) the partial derivatives of h of orders $1,2,\dots,r$ and the norm refers to the supremum over $x \in [0,1]^\ell$,

$$\|D_r h\| = \sup_x \|D_r h(x)\| \ .$$

Remark. We could use instead of $[0,1]^\ell$ another standard ℓ-dimensional manifold (e.g. the unit ball in $\mathbb{R}^\ell$ or the sphere S^ℓ) which would give us an essentially equivalent notion of C^r-size.

3.2. It is obvious that the C^r-size is monotone increasing in r and in $Y \subset \mathbb{R}^n$ and that the C^1-size bounds the diameter and the ℓ-dimensional volume (i.e. the Hausdorff measure) of Y by

$$C^1\text{-size}(Y) \geq \max((\mathrm{Vol}\ Y)^{1/\ell}\ ,\ \ell^{-1/2}\,\mathrm{Diam}\ Y)\ .$$

In fact, if ℓ and r equal one and Y is a smooth ark in $\mathbb{R}^n$, then the C^r-size of Y equals the length of Y . The C^2-size of such a Y measures, in a way, the total curvature of Y but the precise geometric meaning of the C^r-size for $\max(\ell,r) \geq 2$ is rather obscure.

If a subset $Y \subset \mathbb{R}^n$ has C^r-size ≤ 1 and $f : \mathbb{R}^n \to \mathbb{R}^m$ is a C^r-map , then by the chain rule the image Y' of f has

$$C^r\text{-size } Y' \leq \text{const}\ \|D_r f\|\ , \tag{1}$$

for some universal constant depending on r, ,m and n . In fact, (1) remains valid if f is defined on a neighbourhood $U \supset Y$ in $\mathbb{R}^n$ which contains the image of the implied map $h : [0,1]^\ell \to \mathbb{R}^n$. If $C^r\text{-size}(Y) \leq \varepsilon \ell^{-1/2}$, then the ε-neigbourhood U_ε of Y will do.

Every $Y \subset \mathbb{R}^n$ of C^r-size $\leq S$ can be subdivided into j^ℓ subsets of C^r-size $\leq S/j$ for all $j = 1,2...$. This is done by dividing $[0,1]^\ell$ into

j^ℓ-cubes $[0,j^{-1}]^\ell$ and using the obvious scaling map $[0,1]^\ell \to [0,j^{-1}]^\ell$ for each cube of this subdivision.

3.3. ALGEBRAIC LEMMA. *Let* $Y \subset [0,1]^n \subset \mathbb{R}^n$ *be the zero set of (a system of some) polynomials* $p_1,\ldots,p_k$ *on* $[0,1]^n$ *, such that* $\dim Y = \ell$ *. For each* $r = 1,2,\ldots$ *there exists an integer* N_0 *which only depends on* n,r *and* $\deg Y \overset{\text{def}}{=} \sum_{i=1}^{k} \deg p_i$ *, and* C^r*-maps* $h_\nu : [0,1]^\ell \to Y$ *for* $\nu = 1,\ldots,N_0$ *, whose images cover all of* Y *and* $\|D_r h_\nu\| \le 1$ *for* $\nu = 1,\ldots,N_0$ *.* Furthermore,

(i) *each* h_ν *is algebraic of degree* $\le d'$ *for same* d' *depending only on* r *,* $\deg Y$ *and* n *(i.e. the graph of* h *in* $[0,1]^\ell \times \mathbb{R}^n$ *is given by some polynomials of total degree* $\le d'$*) ;*

(ii) *each* h_ν *is a real analytic diffeomorphism of the interior of* $[0,1]^\ell$ *onto its image and these images only meet at the boundaries of the cubes. That is, if* $h_\nu(x) = h_{\nu'}(y)$ *, then* x *and* y *lie in the boundary of* $[0,1]^\ell$ *for all* ν *and* $\nu' = 1,\ldots,N_0$ *.*

The proof is given in 4. To get some insight the reader may look at the hyperbola $xy = \varepsilon$ in the square $\{0 \le x \le 1 \,,\, 0 \le y \le 1\} \subset \mathbb{R}^2$ for small positive ε , say $\varepsilon = 0.0001$ and find h_ν for $r = 2$ and $N = 6$.

3.4. MAIN LEMMA. *Let* Y *be an arbitrary subset in the graph* $\Gamma_g \subset \mathbb{R}^{\ell+m} \supset [0,1]^\ell \times \mathbb{R}^m$ *of a* C^r*-map* $g : [0,1]^\ell \to \mathbb{R}^m$ *and take some positive number* $\varepsilon \le 1$ *. Then* Y *can be subdivided into* $N \le C\varepsilon^{-\ell}(1+\|\partial_r g\|)^{\ell/r}$ *subsets of* C^r*-size* $\le C\varepsilon \operatorname{Diam} Y$ *, where* $\partial_r g$ *denotes the vector assembled of the partial derivatives of* g *of order* r *and where* $C = C(\ell,m,r)$ *is a universal constant.*

PROOF. With a change $g(x) \to ag(\lambda x)+b$ we can make $Y \subset [0,1]^\ell \times [1/3,2/3]^m$ and we also can assume $\operatorname{Diam} Y = 1$. Then, using subdivisions of subsets of C^r-size ≤ 1 to j^ℓ pieces of C^r-size $\le j^{-1}$, we reduce further to the case, where $\varepsilon = 1$. Now, fix a small $\delta > 0$, say $\delta = (m+\ell+r)^{-(m+\ell+r)}$ and let k be the first integer $\ge \delta^{-1}\|\partial_r g\|^{1/r}$. Then cover $[0,1]^\ell$ by k^ℓ images of affine maps $\lambda_\nu : [0,1]^\ell \to [0,1]^\ell$ of the form $\lambda_\nu(x) = k^{-1}x + a_\nu$ for $\nu = 1,\ldots,k$. The composed maps $\lambda_\nu \circ g : [0,1]^\ell \to \mathbb{R}$ have $\|\partial_r(\lambda_\nu \circ g)\| \le k^r \|\partial_r g\|$. Using this we reduce the lemma to the case where , $\|\partial_r g\| \le \delta^r$. (Notice that exactly at this stage we gain a lot for large r) .

Now, we invoke the Taylor polynomial of g of degree $r-1$ at some point $x_0 \in [0,1]$. That is a polynomial map $p : [0,1]^\ell \to \mathbb{R}^m$ of degree (of each component of p) $r-1$ which satisfies, for $\|\partial_r g\| \le \delta^r$ and small δ , by Taylor remainder theorem,

$$||\partial_i (p-g)|| \leq 1/3 \quad \text{for} \quad i = 0,1,\dots,r \ .$$

Then we apply Algebraic Lemma to the part Y_o of te graph of p lying in the unit cube $[0,1]^{\ell+m}$ and get N_o maps $h_\nu : [0,1]^\ell \to [0,1]^\ell \times [0,1]^m$ with $||D_r h_\nu|| \leq 1$ which cover Y_o. Denote by $\bar{h}_\nu$ and $\tilde{h}_\nu$ the $[0,1]^\ell$ - and $[0,1]^m$-components of h_ν correspondingly and observe that $\tilde{h}_\nu = p \circ \bar{h}$ for $\operatorname{Im} h_\nu \subset \Gamma_p$. Then we replace $h_\nu = (\bar{h}_\nu, p \circ \bar{h}_\nu)$ by $h'_\nu = (\bar{h}_\nu, g \circ \bar{h}_\nu)$. Since $|| p-g|| \leq 1/3$, the images of h' contain our Y. Finally, we estimate $D_r h'_\nu$ by

$$\| D_r h'_\nu \| \leq \|D_r h_\nu\| + \|D_r (h_\nu - h'_\nu)\| \leq$$

$$\leq 1 + \| D_r((p-g)\circ \bar{h}_\nu)\| \ .$$

Since $||D_r \bar{h}_\nu|| \leq 1$ and $||D_r (p-g)|| \leq r/3$, we obtain with the chain rule,

$$|| D_r h' || < C(\ell,m,r) \ ,$$

which is the required bound on the C^r-size of the images of h'_ν, $\nu = 1,\dots,N_o$, covering Y. Q.E.D.

3.5. MAIN COROLLARY. *Take an open subset* $U \subset \mathbb{R}^m$, *let* $f : U \to \mathbb{R}^m$ *be a* C^r*-map and let* $Y_o \subset U$ *be a subset of* C^r*-size* ≤ 1 *and such that* Y_o *is far from the boundary* ∂U *of* U. *Namely,* $\operatorname{dist}(Y_o, \partial U) \geq \sqrt{\ell}$. *Then the intersection* Y_1 *of the image* $f(Y_o) \subset \mathbb{R}^m$ *with every cube* $\square \subset \mathbb{R}^m$ *of unit size (i.e. with diameter* $\sqrt{m}$*) can be subdivided into* $N \leq C' || D_r f ||^{\ell/r} + 1$ *subsets of* C^r*-size* ≤ 1 *for some constant* $C' = C'(\ell,m,r)$.

PROOF. Let $h : [0,1]^\ell \to \mathbb{R}^m$ be the map with $||D_r h|| \leq 1$ covering Y_o. By the chain rule, the composed map $g = f \circ h$ has $||D_r g|| \leq C''(\ell,m,r) || D_r f||$ and the Main Lemma applies to $Y = \Gamma_g \cap ([0,1]^\ell \times \square) \subset [0,1]^\ell \times \mathbb{R}^m$. Since Y maps onto Y_1 under the projection $[0,1]^\ell \times \mathbb{R}^m \to \mathbb{R}^m$, the covering of Y by subsets of C^r-size ≤ 1 (unsured by the lemma) induces the required covering of Y_1. Q.E.D.

Remark. An important special case is that of a *linear* map f which, in fact, is sufficient for the proof of Yomdin theorem.

3.6. Suppose, the map f sends U into itself and such that $\operatorname{dist}(f(U), \partial U) \geq \sqrt{\ell}$. Then 3.5. also applies to the pieces of Y_1 of C^r-size ≤ 1 which are provided by 3.5. Then by induction on $i = 1,2,\dots$ we come to the following conclusion.

Let $\square_1, \dots, \square_i$ *be arbitrary unit cubes in* U, *let* $\square'_i$ *denote the pullback of* $\square_i$ *under the* i*-th iterate* f^i *of* f *and* Y_i *be the* f^i *image of the intersection* $Y_o \cap \square'_1 \cap \square'_2 \cap \dots \cap \square'_i$. *Then* Y_i *can be subdivided into* $N_i \leq (C' || D_r f||^{\ell/r} + 1)^i$ *subsets of* C^r*-size* ≤ 1. *In particular*

$$\operatorname{Vol} Y_i \leq (C' || D_r f||^{\ell/r} + 1)^i \ . \qquad (*)$$

3.7. *A bound for* $\mathrm{Vol}\, f^j(Y_o)$. Let Π be the restriction of the standard cubical partition Π_{st} of $\mathbb{R}$ to the above U . Then one has with (*) and the notations in 1.1.,

$$i^{-1}\log \mathrm{Vol} f^i(Y_o) \leq \mathrm{ent}(\Pi|Y_o;f,i) + \ell/r \log\|D_r f\| + c \qquad (**)$$

for some $c = c(\ell,m,r)$.

3.8. The proof of Yomdin theorem

First observe that it suffices to consider the case of maps $f : U \to U$ satisfying the assumption in 3.6. because every manifolds X embeds into some $\mathbb{R}^m$ and every map $X \to X$ extends to the normal neighbourhood $U \subset \mathbb{R}^m$ of X with the normal projection $U \to X$. Furthermore, by scaling U to a larger set $\lambda_o U$ for some $\lambda_o \geq 1$ one can make $\mathrm{dist}(X,\partial U)$ as large as one wishes.

Next consider (rescaled) maps $f_j : jU \to jU$ for $j = 1,2,\ldots,$ defined by $f_j(x) = jf(jx)$ and notice that

(i) $\|\partial_r f_j\| = j^{-r}\|\partial_r f\|$;

(ii) the partition Π of jU into unit cubes corresponds to the partition $\Pi(j)$ of U into j^{-1}-cubes.

(iii) the set jY_o can be subdivided into j^ℓ subsets of C^r-size ≤ 1 .

Now, by the definition of $\mathrm{ent}\, f|Y_o$, for every $\varepsilon > 0$ there exist an integer k , such that

$$\mathrm{ent}(\Pi(j)|Y_o;f^k,i) \leq k\, \mathrm{ent}\, f|Y_o + k\varepsilon$$

for all j and all sufficiently large (depending on j^ℓ and k) i . This is equivalent to

$$\mathrm{ent}(\Pi|jY_o;f_j^k,i) \leq k\, \mathrm{ent}\, f|Y_o + k\varepsilon\ . \qquad (***)$$

Next, we choose j sufficiently large in order to make

$$\|D_r f_j^k\| \leq (1+\varepsilon)\|Df^k\|$$

which is possible by (i). Then we apply (**) to f^k and the j^ℓ pieces of jY_o of C^r-size ≤ 1 (see (iii)) and conclude that

$$i^{-1}\log j^{-\ell}\mathrm{Vol} f^{ki}(Y_o) \leq k\, \mathrm{ent}\, f|Y_o + \ell/r \log\|Df^k\| + k\varepsilon(1+\tfrac{\ell}{r}) + c\ ,$$

for all sufficiently large i . We make $i \to \infty$ and observe that

$$\limsup_{i\to\infty} i^{-1}\log \mathrm{Vol}\, f^{ki}(Y) = k \limsup_{i\to\infty} i^{-1}\log \mathrm{Vol} f^i(Y)$$

for all compact submanifolds $Y \subset X$. Therefore

$$\limsup_{i\to\infty} i^{-1}\log \mathrm{Vol}\, f^i(Y_o) \leq \mathrm{ent}\, f|Y_o + \frac{\ell}{kr}\log\|Df^k\| + \varepsilon(1+\frac{\ell}{r}) + c/k .$$

Then we let $k \to \infty$ and $\varepsilon \to 0$ and obtain,

$$\limsup_{i\to\infty} i^{-1} \log \mathrm{Vol}\, f^i(Y_o) \leq \mathrm{ent}\, f\, Y_o + \ell/r \log^+ \mathrm{Rad}\, Df ,$$

for all subsets $Y_o \subset X$ with C^r-size ≤ 1. Since every compact ℓ-dimensional submanifold Y can be covered by finitely many pieces with C^r-size ≤ 1 this inequality holds true for all Y.

Now, to prove Yomdin inequality (*) in 2.2. with the volume of the graphs $\Gamma_{f^i}|Y$ instead of the images $f^i(Y)$ (we used graphs rahter than images mainly to avoid the multiplicity problem for non-injective maps) we observe that $\Gamma_{f^i}|Y = F^i(\Gamma_{Id}|Y)$ for $F : (y,x) \to (Y,f(x))$ and that $\mathrm{ent}\, f|Y = \mathrm{ent}\, f|(\Gamma_{id}|Y)$. Hence, the above inequality for F in place of f yields Yomdin's (*) for f. Q.E.D.

3.9. C^r-<u>entropy and semicontinuity</u>

Let $g_o, g_1, \ldots, g_i : [0,1]^\ell \to \mathbb{R}^m$ be C^r-maps. Then a collection of maps $h_1, \ldots, h_N : [0,1]^\ell \to [0,1]^\ell$ whose images cover $[0,1]^\ell$ is called an *ε-cover* if $\|D_r h_\nu\| \leq \varepsilon$ and $\|D_r(g_j \circ h_\nu)\| \leq \varepsilon$ for all $j = 0,\ldots,i$ and $\nu = 1,\ldots,N$. Let $\mathrm{ent}_\varepsilon(g_o,\ldots,g_i) = \log N$ for the minimal N for which an ε-cover exists. Observe that

$$\mathrm{ent}_\varepsilon \leq \mathrm{ent}_\delta \leq k^\ell \mathrm{ent}_\varepsilon$$

for $k^{-1}\varepsilon \leq \delta \leq \varepsilon$ and all $k = 1,2,\ldots$.

Next, if $\{h_\nu\}$ is an ε-cover for $g_o \ldots g_i$ and $\{h_{\mu\nu}\}$ is an ε-cover for the composed maps $g_j \circ h_\nu$ for $j = 1,\ldots,i$, $\nu = 1,\ldots,N$, then $\{h_\nu \circ h_{\mu\nu}\}$ also is an ε-cover for $g_o \ldots g_i$, provided $\varepsilon \leq \varepsilon_o$, where $\varepsilon_o = \varepsilon_o(\ell,m,r) > 0$ is a universal constant.

Now let $f : X \to X$ be a C^r-map of a smooth compact submanifold $X \subset \mathbb{R}^m$ and let $g : [0,1]^\ell \to X$ be C^r-smooth. Then the limit

$$\limsup_{i\to\infty} i^{-1}\, \mathrm{ent}_\varepsilon(g, f\circ g, \ldots, f^i g)$$

does not depend on $\varepsilon > 0$ by the earlier discussion and is called *C^r-entropy* $\mathrm{ent}^r(f|g)$. Obviously,

$$\mathrm{ent}^r(f^k|g) = k\ \mathrm{ent}^r(f|g)$$

for all $k = 1,2,\ldots$ and

$$\operatorname{ent}^r(f|g) \geq \operatorname{ent} f|g([0,1]^\ell) \quad ,$$

for all $r = 1,2,\ldots$

Then let

$$\operatorname{ent}_\varepsilon(f,i) = \sup i^{-1} \operatorname{ent}_\varepsilon(g, f\circ g, \ldots f^i\circ g)$$

over all g with $\|D_r g\| \leq 1$. If $\varepsilon \leq \varepsilon_0$ for the above $\varepsilon_0 = \varepsilon_0(\ell,m,r)$, then obviously

$$\operatorname{ent}_\varepsilon(f,i+j) \leq (i+j)^{-1}(i \operatorname{ent}_\varepsilon(f,i) + j \operatorname{ent}_\varepsilon(f,j)) \quad .$$

for all $i,j = 1,2,\ldots$. Therefore, there exists a limit

$$\operatorname{ent}^{r,\ell}(f) = \lim_{i\to\infty} \operatorname{ent}_\varepsilon(f,i)$$

for $\varepsilon \leq \varepsilon_0$ which does not depend on ε and which is semicontinuous in f.

If f_τ *is* C^r*-continuous in* $\tau \in [0,1]$, *then*

$$\limsup_{\tau\to 0} \operatorname{ent}^{r,\ell} f_\tau \leq \operatorname{ent}^{r,\ell} f_0 \ .$$

Also observe that

$$\operatorname{ent}^{r,\ell}(f) \geq \sup_g \operatorname{ent}^r(f|g)$$

over all C^r-maps $g : [0,1]^\ell \to X$.

Remark. There is the following topological version of $\operatorname{ent}^{r\ell}$. Take all $Y \subset X$ with C^r-size ≤ 1 , set

$$s(j;k,i) = \sup_Y \operatorname{ent}(\Pi(j)|Y;f^k,i)$$

(compare 1.2) and define

$$\operatorname{top}_r^\ell f = \liminf_{k\to\infty} \lim_{j\to\infty} \limsup_{i\to\infty} s(j;k,i) \ .$$

Clearly

$$\operatorname{top}_r^\ell f \geq \sup_Y \operatorname{ent} f|Y$$

over all C^r-submanifolds Y of dimension ℓ in X and

$$\operatorname{top}_r^\ell f \leq \operatorname{top}_r^n f = \operatorname{ent} f$$

for all $r = 1,\ldots,$ and $\ell \leq n = \dim X$.

Now by applying the argument in sections 3.4-3.8 to the C^r-entropy directly (without passing to volumes) one sees that

$$\operatorname{ent}^r(f|g) \leq \operatorname{ent}(f|g([0,1]^\ell) + \frac{\ell}{r}\log^+ \operatorname{Rad} Df$$

for all C^r-maps $g : [0,1]^\ell \to X$ and

$$\mathrm{ent}^{r,\ell} f \le \mathrm{top}_r^\ell f + \frac{\ell}{r} \log^+ \mathrm{Rad}\, Df$$

In particular, if f is C^∞-smooth, then

$$\mathrm{ent}\, f = \lim_{r\to\infty} \mathrm{ent}^{r,n} X$$

for $n = \dim X$ and the semicontinuity of $\mathrm{ent}^{r,n}$ implies that of $\mathrm{ent}\, f$.

4. THE PROOF OF ALGEBRAIC LEMMA

4.1. First we prove the lemma for algebraic curves Y in the (x,y)-plane such that the projection of Y to the x-axes is finite-to-one. Such a Y can be obviously divided into $N \le d^4$ segments whose projections to the x-axes are one-to-one. Thus we reduce to the case where Y is the graph of a single valued function $y = y(x)$ for $x \in [0,1]$, such that $\|y(x)\| = \sup_x |y(x)| \le 1$.

Next, we subdivide $[0,1]$ into smaller segments by the points where the derivative y' of y equals ± 1. We switch the roles of x and y at those segments where $|y'| \ge 1$ and reduce the lemma to the case of functions $y = y(x)$, such that $\|y'\| \le 1$. This proves the Lemma for $r=1$ since the map $x \mapsto (x,y(x))$ sends $[0,1]$ into Y with $\|D_1\| \le \sqrt{2}$ and an obvious subdivision into two 1/2-subintervals makes $\|D_1\| \le 1$.

Now, for $r \ge 2$, we assume,

$$\|y'\| \le 1 , \ \|y''\| \le 1, \dots , \|y^{(r-1)}\| \le 1$$

and divide $[0,1]$ by the zero points of the derivative $y^{(r+1)}(x)$. Then $y^{(r)}(x)$ is monotone on every subinterval (where $y^{(r+1)}$ does not change sign) and the problem obviously reduces to the case where $y^{(r)}(x)$ is positive and monotone decreasing on $[0,1]$. This monotonicity and the bound $\|y^{(r-1)}\| \le 1$ imply that $y^{(r)}(x) \le 2x^{-1}$ for all $x \in [0,1]$. Then a straightforward computation shows that the function $z(x) = y(x^2)$ has

$$\|z^{(i)}\| \le 10^r \quad \text{for } i = 1,\dots,r ,$$

and the map $x \to (x,z(x))$ with an additional subdivision into 10^r segments provides the proof of Algebraic lemma for plane curves Y.

4.2. Now, let Y be a curve in $[0,1]^n \subset \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$. We may assume that the projection of Y to $\mathbb{R}$ is finite to one. Then Y is the graph of $(n-1)$ algebraic functions $y_1(x)$, $y_2(x)$ ... $y_{n-1}(x)$. We assume, by induction, that the functions $y_1,\dots,y_{n-2}$ have bounded derivatives of orders $\le r$ and use the above change of variable, $x \leftrightarrow x(t)$ to make the derivatives of y_{n-1} also bounded. Then all functions $z_i(t) = y_i(x(t))$, $i = 1,\dots,n-1$ have bounded derivatives (on some subintervals) which obviously yields Algebraic Lemma for Y.

4.3. Consider a smooth vector valued algebraic function in ℓ variables, say $y = y(x_1,\dots,x_\ell)$, such that the components of the partial derivatives of orders $\leq r$ in the first $\ell-1$ variables are bounded in absolute values by one and let us make a change in the variable x_ℓ in order to achieve a similar bound for all partial derivatives. We assume by induction on r that the partial derivatives of orders $\leq s \leq r$ in x_ℓ are bounded. We denote by $\tilde{y} = \tilde{y}(x_1,\dots,x_\ell)$ the vector valued function whose components are the partial derivatives of the orders $\leq i_j$ in x_j , where

$$\sum_{j=1}^{\ell} i_j = r \quad \text{and} \quad i_\ell \leq s .$$

Let $\tilde{y}_1,\dots,\tilde{y}_N$ be the components of $\tilde{y}$ and assume by induction on the number of components that

$$\left\|\frac{\partial \tilde{y}}{\partial x_\ell}\right\| \leq 1 \quad \text{for} \quad \nu = 1,\dots,M-1 < N .$$

Then, for every fixed value of $x_\ell \in [0,1]$ we consider the maximum set $S(x_\ell) \subset x_\ell \times [0,1]^{\ell-1} = [0,1]^{\ell-1}$ of the function $\left|\frac{\partial \tilde{y}_M}{\partial x_\ell}\right|$ in the variables $x_1,\dots,x_{\ell-1}$. Then there obviously exists a subdivision of $[0,1]$ into subintervals, say I_k , and single valued algebraic functions $s_k : I_k \to [0,1]^{\ell-1}$, such that

(a) the number of the subintervals and $\deg s_k$ are bounded in terms of $\deg \tilde{y}_M$;

(b) $s_k(x_\ell) \in S(x_\ell)$ for all k and all $x_\ell \in I_k$.

Define $\tilde{s}_k : I_k \to [0,1]^{\ell-1} \times [-1,1]$ by $\tilde{s}_k : x_\ell \mapsto (s_k(x_\ell)\ ,\ \tilde{y}_M(x(x_\ell))$ and apply the construction of the previous section to each function $\tilde{s}_k(x_\ell)$. This makes the derivatives $\frac{d^i \tilde{s}_k(x_\ell)}{d^i x_\ell}$ bounded for $i = 1,\dots,r$ and all k which easily implies a bound on $\frac{\partial \tilde{y}_M}{\partial x_\ell}$.

4.4. Now we prove Algebraic lemma by induction on $\ell = \dim Y$ for an algebraic set $Y \subset [0,1]^n$. We view this Y as the graph of an algebraic map $y : [0,1]^\ell \to [0,1]^{n-\ell}$ and we assume, for every fixed $x_\ell \in [0,1]$, that there exists some change of variables $x_1,\dots,x_{\ell-1}$ providing a universal bound for the partial derivatives of every branch of y in the changed variables $x_1,\dots,x_{\ell-1}$. We asssume, moreover, this change of variables be the piece-wise algebraic in x_ℓ and thus come to the situation of the previous section. Since the constructions we use in 4.1. are piece-wise algebraic for families of functions algebraicly depending on parameters, this induction does go through and the Algebraic Lemma is proven.

4.5. The above argument provides a (semi) algebraic cell decomposition of an arbitrary semi-algebraic set Y and the cells can be (obviously) subdivided into simplices without loosing the control over the partial derivatives, such that the number of the simplices is bounded in terms of $\deg Y$.

Recall, that a subset $Y \subset \mathbb{R}^n$ is called *semialgebraic* if it is a finite union of pairwise non-intersecting subsets $Y_1, \dots Y_k$ in Y where each Y_i is a connected component of the difference of algebraic sets, $Y_i \subset A_i \setminus B_i$. The sum of the degrees of the polynomials defining all A_i and B_i is called the *degree* of Y.

Now we give a precise version of the previous remark.

TRIANGULATION LEMMA. *There exists a constant* $C = C(n,r)$, *such that every compact semialgebraic subset* $Y \subset \mathbb{R}^n$ *can be triangulated into* $N \leq (\operatorname{diam} Y)^n (\deg Y+1)^C$ *simplices, where for every closed* k-*simplex* $\Delta \subset Y$ *there exists a homeomorphism* h_Δ *of the regular simplex* $\Delta^k \subset \mathbb{R}^k$ *with the unit edge length onto* Δ *such that* h_Δ *is algebraic of degree* $\leq (\deg Y+1)^C$ (*i.e. the graph of* h_Δ *is a subset in an algebraic set of dimension* k *and degree* $\leq (\deg Y+1)^C$) *and regular real analytic in the interior of each face of* Δ. ("*Regular*" *means non-vanishing of the differential of* h_Δ *on non-zero vectors*). *Furthermore,* $\|D_r h_\Delta\| \leq 1$ *for all* Δ. (*Of course, just this inequality makes the triangulation truly interesting*).

Using this lemma and the argument in §3 we arrive at the following Corollary.

TRIANGULATION THEOREM. *Let* f *be a* C^r-*selfmap of an open subset* $U \subset \mathbb{R}^n$, *such that* $\|D_r f\| < \infty$ and let $Y \subset U$ *be a compact semialgebraic subset. Then there exists a sequence of triangulation* T_i *of* Y *where* T_{i+1} *is a refinement of* T_{i+1} *for all* $i = 1,2\dots$, *and such that*

(a) *The number* N_i *of simplices of* T_i *satisfies*

$$\limsup_{i\to\infty} i^{-1} \log N_i \leq \operatorname{ent} f|Y + \frac{\ell}{r} \log^+ \operatorname{Rad} Df ,$$

for $\ell = \dim Y$. (*If* Y *is invariant under* f, *then this inequality obviously implies*

$$\log \operatorname{Rad} f_* H_*(Y) \leq \operatorname{ent} f|Y + \frac{\ell}{r} \log^+ \operatorname{Rad} Df) .$$

(b) *For every* k-*simplex* Δ *of* T_i *there exists an algebraic homeomorphism* $h : \Delta^k \to \Delta$ *which has degree* $\leq d_i$ *and satisfies* $\|D_r (f^j \circ h)\| \leq \varepsilon_i$ *for all* $j \leq i$, *where*

$$i^{-1} \log d_i \to 0 \quad \text{for} \quad i \to \infty$$

and $\varepsilon_i \to 0$. (*This and* (a) *sharpen* (**) *in* 2.2. .

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Mikhael GROMOV

Institut des Hautes Etudes Scientifiques
35, route de Chartres
91440 Bures-sur-Yvette