DIFFEOMORPHISMS WITH POSITIVE METRIC ENTROPY

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Abstract. We obtain a dichotomy for C^1-generic, volume preserving diffeomorphisms: either all the Lyapunov exponents of almost every point vanish or the volume is ergodic and non-uniformly Anosov (i.e. hyperbolic and the splitting into stable and unstable spaces is dominated).

We take this dichotomy as a starting point to prove a C^1 version of a conjecture by Pugh and Shub: among partially hyperbolic volume-preserving C^r diffeomorphisms, $r > 1$, the stably ergodic ones are C^1-dense.

To establish these results, we develop new perturbation tools for the C^1 topology: "orbitwise" removal of vanishing Lyapunov exponents, linearization of horseshoes while preserving entropy, and creation of "superblenders" from hyperbolic sets with large entropy.

Contents

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Introduction

The colloquial formulation of Boltzmann's Ergodic Hypothesis posits that "space averages equal time averages" in many systems of physical origin. After George Birkhoff's 1931 proof of the ergodic theorem, this vague property could be rephrased precisely as a statement about the ergodicity of a conservative dynamical system. While the simplest, integrable, dynamical systems are manifestly nonergodic, Birkhoff, E. Hopf and others went on to express the belief that ergodicity is the "general case" [OU].

In 1941, Oxtoby and Ulam showed that among conservative *homeomorphisms* of any compact manifold, ergodicity is indeed a generic property from the topological point of view (i.e., it corresponds to a *residual* subset in the uniform topology on the space of homeomorphisms). Oxtoby and Ulam themselves recognized the limitations of their result as it pertains to the motivating Ergodic Hypothesis, which referred specifically to *smooth* dynamical systems, the diffeomorphisms and flows associated with Hamiltonian mechanics. They wrote: "It must be emphasized, however, that our investigation is on the topological level. The flows we construct are continuous groups of measure-preserving automorphisms, but not necessarily differentiable or derivable from differential equations. Thus they correspond to dynamical systems only in a generalized sense."

The question of whether an Oxtoby-Ulam type result should hold for Hamiltonian systems (or, more generally volume-preserving diffeomorphisms and flows) remained open until the pioneering work of Kolmogorov in the 1950's on what came to be known as Kolmogorov-Arnol'd-Moser (KAM) Theory. In his 1954 ICM address [Ko], Kolmogorov wrote: "Are dynamical systems on compact manifolds 'generally speaking' transitive, and should we consider a continuous spectrum as the 'general' case and a discrete spectrum as the 'exceptional' case? ... In the application to analytic canonical[1] systems, the answer to both questions is negative [...]"

In the analytic setting Kolmogorov had in mind, ergodicity is indeed *not* a dense property in general. More precisely, Kolmogorov showed that for any area-preserving analytic surface diffeomorphism, the existence of a suitably nondegenerate elliptic periodic point is a *robust* obstruction to ergodicity: in a neighborhood of such a point, there is a positive measure set of invariant curves that cannot be perturbed away under any change of analytic parameters. As later shown by Arnol'd, Moser and others, this "stable obstruction to ergodicity" occurs much more generally: on any compact connected manifold M, and for sufficiently large r (depending on the dimension), the space $\mathrm{Diff}^r_m(M)$ of C^r volume-preserving diffeomorphisms contains an open set of non-ergodic diffeomorphisms.

To this day, it is not known whether KAM-type robust obstructions to ergodicity exist for less regular transformations. In particular, the possibility remains that ergodicity is a dense (and hence generic) property in $\mathrm{Diff}^1_m(M)$ or $\mathrm{Diff}^2_m(M)$. So it is natural to ask which of the Oxtoby-Ulam or Kolmogorov scenarios give a better description of diffeomorphisms in low regularity, though in either case, it is certain that the underlying mechanism behind it would have to be rather different.

One of the goals of this paper is to provide an answer to this question among those conservative diffeomorphisms presenting positive *entropy*. The (metric) notion of entropy, introduced by Kolmogorov and Sinai, provides one of the most basic measures of the "visible" chaoticity of a dynamical system: positive entropy corresponds to the exponential growth in complexity of the dynamics from an information theoretical point of view. Such an assumption turns out to play a key role in the analysis of a wide variety of dynamical phenomena, both in the classical setting discussed here [Ka], as in the broader context of general group actions [EKL].

In this respect, it is notable that the Oxtoby-Ulam construction, based on periodic approximation, implies that generic conservative homeomorphisms have zero metric entropy. In contrast, positive metric entropy appears robustly in $\mathrm{Diff}^1_m(M)$, as can be seen most easily in the case of *Anosov maps*. Interestingly, the *uniformly hyperbolic* behavior that gives rise to positive metric entropy in Anosov systems is also the source of a powerful mechanism for ergodicity, of a very different nature than the Oxtoby Ulam mechanism.

We will show that positive metric entropy is associated with *non-uniformly hyperbolic* behavior for generic diffeomorphisms in $\mathrm{Diff}^1_m(M)$. This is enough for us to conclude:

Theorem A. *A generic map $f \in \mathrm{Diff}^1_m(M)$ with positive metric entropy is ergodic.*

Entropy and non-uniform hyperbolicity. The Kolmogorov-Sinai entropy of a volume preserving diffeomorphism is closely connected to a somewhat finer measurement of chaoticity which is given by the notion of Lyapunov exponents. A real

[1] i.e., Hamiltonian.

number χ is a *Lyapunov exponent* of a diffeomorphism $f : M \to M$ at $x \in M$ if there exists a nonzero vector $v \in T_xM$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|Df^n(v)\| = \chi. \tag{1}$$

If f preserves the volume m, then Kingman's ergodic theorem implies that the limit $\chi = \chi(x, v)$ in (1) exists for for m-almost every $x \in M$ and every nonzero $v \in T_xM$. Furthermore, Oseledets's theorem gives that $\chi(x, \cdot)$ can assume at most $\dim(M)$ distinct values $\chi_1(x), \dots, \chi_{\ell(x)}(x)$, and that there exists a measurable, Df-invariant splitting of the tangent bundle

$$TM = E_1 \oplus E_2 \oplus \cdots \oplus E_\ell, \tag{2}$$

such that $\chi(x, v) = \chi_i(x)$ for $v \in E_i(x) \setminus \{0\}$. Volume preservation implies that the sum of the Lyapunov exponents is zero.

Entropy and Lyapunov exponents of C^1 diffeomorphisms are related by Ruelle's inequality, which states that for $f \in \mathrm{Diff}^1(M)$ preserving a probability measure μ:

$$h_\mu(f) \le \int_M \sum_{\chi_i(x)\ge 0} \dim(E_i(x))\chi_i(x)\, d\mu(x). \tag{3}$$

For for $\mu = m$, the reverse equality was proved by Pesin for all $f \in \mathrm{Diff}^2_m(M)$ and C^1-generically in $\mathrm{Diff}^1_m(M)$ by Tahzibi [T1], Sun-Tian [ST].

In his 1983 address at the ICM [M], Mañé proposed to study how the "Oseledets splitting" (2) varies as a function of the diffeomorphism f, in the C^1 topology. He announced the following remarkable result, whose proof was later completed by Bochi [Boc1]. For the generic surface diffeomorphism $f \in \mathrm{Diff}^1_m(S)$,

– either the Lyapunov exponents $\chi_i(x)$ of f vanish for m-a.e. x,
– or f is *Anosov*: there exists a continuous, Df-invariant splitting

$$TM = E^u \oplus E^s \tag{4}$$

and constants $0 < \lambda < 1$, $n_0 \in \mathbb{N}$, such that $\|Df^n|E^u\| \le \lambda^n$ and $\|(Df^n|E^s)^{-1}\| \le \lambda^n$ for every $n \ge n_0$.

In the second, *uniformly hyperbolic* case, the Oseledets splitting (2) coincides with the Anosov splitting (4), the Lyapunov exponents $-\chi < 0 < \chi$ are nonzero constants, and the number λ can be chosen arbitrarily close to $\exp(-\chi)$.

Note that by Ruelle's inequality, maps with only zero Lyapunov exponents have zero entropy. While it is still unknown (even in dimension two) whether all (conservative) Anosov diffeomorphisms are ergodic, it is certainly the case for more regular (C^2) maps, and an approximation argument allows one to conclude that C^1-generic Anosov diffeomorphisms are ergodic. So the Mañé-Bochi theorem implies Theorem A in dimension two.

The seemingly immediate approximation argument described above, due to Zehnder in the case of surfaces, turned out to be considerably more delicate in general, having been established only recently by Avila [Av]. It thus became reasonable to approach Theorem A through a suitable dichotomy "zero entropy or hyperbolicity" in the spirit of the Mañé-Bochi theorem.

However, in higher dimensions, it was understood that uniform hyperbolicity was too much to aim for. Indeed any conservative dynamics admitting a *dominated splitting* must have robustly positive metric entropy. A diffeomorphism

$f \in \mathrm{Diff}(M)$ is said to admit a (global) dominated splitting if there exists a continuous non-trivial decomposition $TM = E_1 \oplus E_2$ that is Df-invariant and satisfies $\|Df^N|E_1\|\|(Df^N|E_2)^{-1}\| < 1$ for some $N \in \mathbb{N}$. Thus f is an Anosov map if and only if it admits a uniformly hyperbolic dominated splitting.

While in dimension 2 a dominated splitting for a conservative diffeomorphism is always uniformly hyperbolic (i.e. Anosov), already in dimension 3 there are manifolds which do not support Anosov dynamics, but which are compatible with a dominated splitting. In the presence of robust obstructions to uniform hyperbolicity, the best one can hope for is to obtain a dominated splitting $TM = E^+ \oplus E^-$ that is *non-uniformly hyperbolic*, in the sense that there exists $\chi_0 > 0$ such that for m-a.e. $x \in M$, if $v \in E^+(x) \setminus \{0\}$, then $\chi(x,v) > \chi_0$, and if if $v \in E^-(x) \setminus \{0\}$, then $\chi(x,v) < -\chi_0$.

A map f admitting a non-uniformly hyperbolic dominated splitting will be called *non-uniformly Anosov*. Equivalently, f is non-uniformly Anosov if it possesses a dominated splitting $TM = E^+ \oplus E^-$ and there exists $0 < \lambda < 1$ such that for m-almost every $x \in M$, there exists $n_0(x) \in \mathbb{N}$ such that $\|Df^n(x)|E^-(x)\| \leq \lambda^n$ for every $n \geq n_0(x)$.

Theorem A'. *For a generic map $f \in \mathrm{Diff}^1_m(M)$, either*

(1) *the Lyapunov exponents of f vanish almost everywhere, or*
(2) *f is non-uniformly Anosov and ergodic.*

This result was conjectured by Avila-Bochi [AB] where it was shown that generic diffeomorphisms in $\mathrm{Diff}^1_m(M)$ with only non-zero Lyapunov exponents almost everywhere are ergodic and non-uniformly Anosov. Notice that by Ruelle's inequality, Theorem A follows immediately from Theorem A'.

In dimension three, Theorem A' was proved by M.A. Rodríguez-Hertz [R] by reducing to an analysis of dominated splittings admitting some uniformly hyperbolic subbundles, which have been thoroughly described for 3-manifolds. Some other approaches to ergodicity based on a strengthening of the notion of dominated splitting, *partial hyperbolicity*, will be recalled in the next section. Our proof of Theorem A' in the general case follows a very different route, focused on the elimination of zero Lyapunov exponents throughout large parts of the phase space.

The existence of a dominated splitting is a robust dynamical property (i.e., stable under perturbations in $\mathrm{Diff}^1(M)$), as is uniform hyperbolicity. A striking consequence of Theorem A' is thus:

Corollary. *A map $f \in \mathrm{Diff}^1_m(M)$ has robust positive metric entropy if and only if it admits a dominated splitting.*

These results highlight the unique features of the C^1 topology. At least conjecturally, sufficiently regular volume preserving diffeomorphisms are expected to be compatible with a quite different phenomenon: the coexistence of quasiperiodic behavior (where Lyapunov exponents vanish) with chaotic, non-uniformly hyperbolic behavior. Even on surfaces, this conjecture remains open.

Elimination of zero Lyapunov exponents. From the development of Pesin Theory and the gradual taming of (sufficiently regular) non-uniformly hyperbolic dynamics which followed ([P], [Ka]), it has been a central problem to understand how often such systems arise. While it is understood that the "opposite" behavior, the vanishing of all Lyapunov exponents, does appear robustly (through the KAM mechanism), it has been proposed by Shub and Wilkinson ([SW], Question 1a) that for typical orbits of a generic C^r conservative dynamical system, the presence of some non-zero Lyapunov exponent implies in fact that all Lyapunov exponents are non-zero. Such an optimistic picture was motivated by an argument, introduced in the same paper, which allows one to leverage (in a particularly controlled setting) the non-zero Lyapunov exponents to "perturb away" the zero Lyapunov exponents.

The specific situation considered by Shub and Wilkinson consisted of a trivial circle extension of a linear Anosov map. This is a partially hyperbolic dynamical system with a one-dimensional central direction along which the Lyapunov exponent vanishes everywhere. Through a carefully designed perturbation, the central bundle borrows some of the hyperbolicity from the uniformly expanding bundle, so the *average* central Lyapunov exponent becomes positive. In order to show that the actual Lyapunov exponent along the center is non-zero almost everywhere, they observe that the system can be, at the same time, made ergodic by a separate argument (based on the Pugh-Shub ergodicity mechanism).

This argument has been pursued further, in low regularity, by Baraviera and Bonatti [BB]. They consider conservative diffeomorphisms admitting a dominated splitting $TM = E_1 \oplus \cdots \oplus E_k$ and show that the average of the sum of the Lyapunov exponents along any subbundle can be made non-zero by a C^1 perturbation. This result was used by Bochi, Fayad and Pujals in [BFP] to show that stably ergodic diffeomorphisms, which admit a dominated splitting by [BDP], can be made non-uniformly hyperbolic by perturbation.

In a sense, here we do just the opposite of [BFP]: we show the generic absence of zero Lyapunov exponents almost everywhere (under the positive entropy assumption) as a means to conclude ergodicity (via [AB]). In order to do this, we must develop a perturbation argument that can affect directly the actual Lyapunov exponents of certain orbits, and not just their averages.

We remark that the existence of a global dominated splitting, which is a starting point in [SW] and [BB], is here also obtained as a consequence of non-uniform hyperbolicity (again, via [AB]). The hypothesis of positive entropy (and hence the existence of some non-zero Lyapunov exponents) is however enough to obtain local dominated splittings, thanks to a result of Bochi and Viana [BV2] who showed that, for almost every orbit of generic conservative diffeomorphisms, the Oseledets splitting extends continuously to a dominated splitting on its closure.

Our basic technique is the following. First, we may assume that the initial (generic) diffeomorphism has a positive measure set of orbits having some, but not all, non-zero Lyapunov exponents (otherwise [AB] yields the conclusion at once). Consider a sufficiently long segment of a typical orbit that admits a dominated splitting $E_1 \oplus E_2 \oplus E_3$, where E_2 corresponds to zero Lyapunov exponents. If this orbit segment is long enough, then it "sees" the Lyapunov exponents of the orbit. We can then reproduce the perturbation technique of [SW] and [BB] along the orbit: since this technique concerns average exponents, we first thicken the initial point to a small positive measure set, and conclude that the average of the sum of the Lyapunov exponents along the central bundle can be decreased. In order to produce a pointwise estimate, we use a randomization technique introduced by Bochi in [Boc2], which allows us to apply the Law of Large Numbers to promote the averaged estimate to a pointwise one. Using a standard towers technique, this argument can be carried out simultaneously a large set of the orbits remaining within the domain of definition U of the local dominated splitting.

Naturally, the perturbation changes the dynamics, so in principle the decrease of the sum of Lyapunov exponents could be cancelled later. In fact the dynamics could change so much that many orbits escape U and we lose all control, but this "loss of mass" is an irreversible event and thus relatively harmless. As for possible cancellations, we simply assume away the problem by restricting attention to the case where the Lyapunov exponents along E_2 are non-positive for almost every orbit that remains within U. Remarkably, this seemingly very strong hypothesis can be in fact verified along the steps of a carefully designed inductive argument. In any case, with this assumption we can conclude directly that for most orbits remaining in U the number of zero Lyapunov exponents is strictly less than the dimension of E_2, after perturbation.

Iterating this argument, we eventually succeed in either eliminating all non-zero Lyapunov exponents, or in obtaining vanishing Lyapunov exponents almost everywhere (this happens when we keep running into the situation where orbits escape the domains of definition of local dominated splittings).

Stable ergodicity. While the development of KAM theory effectively demolished hope that an Ergodic Hypothesis in its most general formulation could hold for analytic (or even C^∞) conservative systems, at the same time the notion of a refined Ergodic Hypothesis in the setting of higher regularity began to take hold. In the same 1954 address [Ko], Kolmogorov wrote: "it is extremely likely that, for arbitrary k, there are examples of canonical systems with k degrees of freedom and with stable transitiveness and mixing... I have in mind motion along geodesics on compact manifolds of constant negative curvature... " With the Hopf ergodicity results for geodesic flows in mind, Kolmogorov was in essence positing the existence of a larger, open class of ergodic Hamiltonian flows, containing the geodesic flows as a very special subclass. Kolmogorov's intuition was clearly guided by the robust nature of Hopf's ergodicity proof; indeed, save for the single technical obstruction of nonsmoothness of invariant stable and unstable foliations, Hopf's approach gives a complete proof of Kolmogorov's assertion.

Around ten years later, Anosov [An1] overcame this obstruction by formalizing the notion of absolute continuity of a foliation and proving that for regular uniformly hyperbolic flows and diffeomorphisms, the stable and unstable foliations are absolutely continuous. More precisely, replacing smoothness of these foliations with absolute continuity in Hopf's original proof, Anosov established that these systems, now called Anosov diffeomorphisms and flows, are always ergodic with respect to an invariant volume, provided they are of class C^2. Such a regularity assumption (which can be replaced by $C^{1+\alpha}$, for some $\alpha > 0$) plays an indispensible role in Anosov's proof of absolute continuity. As a special case of Anosov's result one obtains ergodicity of the geodesic flow on the unit tangent bundle of any closed manifold of strictly negative sectional curvatures.

We should remark that, even today, essentially all proofs of ergodicity in a context where hyperbolicity is present depend to a certain extent on the *Hopf argument*: Any measurable invariant set is essentially saturated by stable (respectively, unstable) manifolds, i.e., it coincides, up to a zero Lebesgue measure set, with a measurable invariant set that is a union of stable (respectively, unstable) manifolds. The way it can be used to deduce ergodicity is particularly simple in the original situation considered by Hopf, since the stable and unstable foliations are smooth and the stable manifold of any orbit intersects transversely the unstable manifold of any other orbit. In the more general, uniformly hyperbolic context (assuming regularity), this intersection property remains true, while the fact that the foliations are no longer smooth but merely absolutely continuous poses no special difficulty in concluding ergodicity. More involved applications of the Hopf argument will be described in the next section.

In the spirit of Kolmogorov's assertion, we can reformulate the ergodicity of this open class of diffeomorphisms and flows as a property of the individual elements. We say that a diffeomorphism $f \in \mathrm{Diff}^2_m(M)$ is *stably (or robustly) ergodic* if any $g \in \mathrm{Diff}^2_m(M)$ which is C^1-close to f is ergodic. One similarly defines stably ergodic flows. Thus Anosov's theorem implies that C^2 volume preserving Anosov diffeomorphisms are stably ergodic. Whether it would be reasonable to drop completely the regularity requirement in the notion of stable ergodicity remains unclear: it is unknown whether there exists any open set in $\mathrm{Diff}^1_m(M)$ (in any manifold M) consisting of ergodic diffeomorphisms.

Since stable ergodicity is by definition a robust property, it is natural to search for an alternate robust, geometric/topological dynamical characterization of the stably ergodic diffeomorphisms. On the one hand, as mentioned above, Bonatti-Diaz-Pujals [BDP] showed that a conservative diffeomorphism lacking a dominated splitting can always be C^1 perturbed to contain a periodic ball, a clear obstruction to ergodicity. Hence the stably ergodic diffeomorphisms are a subset of the diffeomorphisms preserving a dominated splitting, and, it is easy to see, a proper subset.

Oxtoby and Ulam remarked in their 1941 paper that the initial goal of their research was to address the question: does every manifold admit an ergodic, conservative dynamical system? Subsequent to the Oxtoby-Ulam theorem for homeomorphisms, this question has been answered affirmatively for diffeomorphisms in all regularity [BFK, DP]. This puts to rest the question of whether there are topological obstructions to ergodicity. But we remark that when "ergodicity" is replaced by "stable ergodicity" the answer is no, even on surfaces: since stable ergodicity implies a nontrivial dominated splitting, in dimension 2 only the torus can support a stably ergodic diffeomorphism. Hence the original question tackled by Oxtoby and Ulam was not vacuous when viewed through a robust lens.

A C^1-open subclass of the systems with a dominated splitting, and one containing the Anosov diffeomorphisms, is the set of partially hyperbolic diffeomorphisms. One says that $f \in \mathrm{Diff}^1_m(M)$ is partially hyperbolic (with three-way splitting) if there is a nontrivial Df-invariant splitting $TM = E^u \oplus E^c \oplus E^s$ such that $E^u \oplus (E^c \oplus E^s)$ and $(E^u \oplus E^c) \oplus E^s$ are both dominated splittings, and the bundles E^u and E^s are uniformly expanded and contracted, respectively.

Grayson, Pugh and Shub [GPS] established the existence of non-Anosov stably ergodic diffeomorphisms in 1995 by proving stable ergodicity for the time-one map of the geodesic flow for a surface of constant negative curvature.[2] Based on this and related work, Pugh and Shub formulated in 1996 a bold conjecture about stable ergodicity [PS]:

[2]Note that the time one map of an Anosov flow is not an Anosov diffeomorphism, but is partially hyperbolic, with one dimensional center tangent to the vector field generating the flow.

Stable Ergodicity Conjecture. *Stable ergodicity is C^r-dense among the C^r partially hyperbolic volume-preserving diffeomorphisms on a compact connected manifold, for any $r > 1$.*

In the particular case where the center bundle is one-dimensional, this has been proved by F. Rodriguez-Hertz, M.A. Rodriguez-Hertz and Ures [RRU]. Here we will establish, in general, a C^1 version of the Stable Ergodicity Conjecture:

Theorem B. *Stable ergodicity is C^1-dense in the space of C^r partially hyperbolic volume-preserving diffeomorphisms on a compact connected manifold, for any $r > 1$,*

More precisely, this space contains a C^1 open and dense subset of diffeomorphisms which are non-uniformly Anosov, ergodic and in fact Bernoulli.

When the center bundle has dimension one or two, Theorem B has been established earlier in [BMVW] and [RRTU].

Robust mechanisms for ergodicity. The formulation of the Stable Ergodicity Conjecture was supported by a proposed mechanism for ergodicity in partially hyperbolic systems that Pugh-Shub called “accessibility,” a term borrowed from the control theory literature[3] . As is the case with Anosov diffeomorphisms, the unstable bundle E^u and the stable bundle E^s of a partially hyperbolic diffeomorphism are uniquely integrable, tangent to the unstable and stable foliations, $\mathcal{W}^u$ and $\mathcal{W}^s$, respectively. A partially hyperbolic diffeomorphism is *accessible* if any two points in the manifold can be connected by a continuous path that is piecewise tangent to leaves of $\mathcal{W}^u$ and $\mathcal{W}^s$.

The route suggested by Pugh-Shub to prove the Stable Ergodicity Conjecture was to establish two other conjectural statements: 1) stable accessibility is dense, and 2) accessibility implies ergodicity.

The C^r-density of stable accessibility remains open, except in the case of one-dimensional center bundle [RRU]. However, in the C^1 case we are concerned with it was established by Dolgopyat and Wilkinson [DW] in 2003.

The connection between accessibility and ergodicity turns out to be the more difficult issue for us. It is certainly reasonable at the heuristic level: The Hopf argument can be carried out in the partially hyperbolic setting, and the ergodic (and even mixing) properties of the system can be reduced to the ergodic properties of the measurable equivalence relation generated by the pair of foliations $(\mathcal{W}^u, \mathcal{W}^s)$. However, in trying to show that these ergodic properties follow from the assumption of accessibility, one quickly encounters substantial issues involving sets of measure zero and delicate measure-theoretic and geometric properties of the foliations.

Currently these issues can be resolved only under an additional assumption called center bunching (which constrains the nonconformality of the dynamics along the center bundle E^c), a result due to Burns and Wilkinson [BW]. In all likelyhood, a significant new idea is needed to make further progress. And as it turns out, center bunching is certainly not a dense property among partially hyperbolic diffeomorphism, except in the case of one-dimensional center bundle, when it is automatically satisfied.

While ergodicity is difficult to conclude from accessibility, a relatively simple argument (due to Brin) allows one to conclude that accessibility implies *metric*

[3] Brin and Pesin had earlier studied the accessibility property and its stability properties in the 1970’s and used it to prove topological properties of conservative partially hyperbolic systems.

transitivity, that is, that almost every orbit is dense. This motivates the idea of combining accessibility with some "local" ergodicity mechanism in order to achieve full ergodicity. The approach we will follow, due to [RRTU], is in a certain sense closer to the original applications of the Hopf argument than the Pugh-Shub approach, but with one crucial new ingredient added.

In the absence of uniform hyperbolicity, one still has the Pesin stable and unstable disk families, which are also absolutely continuous. The dimension and diameter of these (smooth) Pesin stable disks vary measurably over the manifold. If we want to implement the Hopf argument as Hopf and Anosov did, the main issue is to ensure the transverse intersection of stable and unstable disks. Several difficulties arise:

(1) The Pesin disks should have "enough dimension" so that transversality is even possible.[4]
(2) The Oseledets spaces along which they align should display some definite transversality.
(3) The stable and unstable manifolds should be "long enough" so that they have the opportunity to intersect.

The first two difficulties are resolved more or less automatically in the context of perturbations of an ergodic non-uniformly Anosov map: one can arrange things so that the stable manifolds under consideration align with the space E^- and the unstable manifolds align with the space E^+.

In the context where the diffeomorphism is assumed to be partially hyperbolic, there exist in addition to the Pesin manifolds, strong stable and unstable foliations whose leaves are immersed submanifolds. The leaves of the unstable foliation subfoliate the unstable Pesin manifolds, and similarly in the stable directions. Thus in the partially hyperbolic context, the (Pesin) stable and unstable manifolds are *definitely large* in the strong directions. To address the third difficulty, we need a way of increasing the size of stable and unstable manifolds in non-strong directions to a definite scale.

In [RRTU], it is shown how so-called stable and unstable *blenders* can be used to resolve this third difficulty in a partially hyperbolic context. An unstable blender is a "robustly thick" part of a hyperbolic set, in the sense that its stable manifold meets every strong unstable manifold that comes near it, and moreover this property is still satisfied (by its hyperbolic continuation) after any C^1 perturbation of the dynamics. The key point is that this property may be satisfied even if the dimensions of the strong stable manifolds and strong unstable manifolds are not large: the (thick) fractal geometry of the blender will be responsible for yielding the "missing dimensions."

Since the Pesin unstable manifolds contain the strong unstable manifolds, we conclude that any Pesin unstable manifold near the blender has a part that is trapped by the blender dynamics. Under iteration, this trapped part evolves according to the hyperbolic dynamics of the blender, which enlarges even the non-strong directions to a definite size. Analogously defined stable blenders play a similar role of enlarging the Pesin stable manifolds to a definite size. If the unstable and stable

[4]Note that in the Pugh-Shub approach, the Hopf argument is applied in a context where the dimensions of the strong stable and unstable manifolds are not enough to allow for transversality, but the analysis is much more involved than the simple implementation we are discussing, and it ends up depending on the center bunching condition.

blenders are contained in a larger transitive hyperbolic set, then those long pieces of unstable and stable manifolds do get close to one another and will thus intersect as desired.

How often do blenders arise in the context of partially hyperbolic dynamics? Originally, blenders were constructed using a very concrete geometric model, which was then seen to arise in the unfolding of heterodimensional cycles between periodic orbits whose indices differ by one [BD]. The fractal geometry of such a blender effectively yields one additional dimension in the above argument, so in order to obtain multiple additional directions, one would need to use several such blenders. Unfortunately, there are robust obstructions to the construction of some of the heterodimensional cycles needed to produce such blenders.

A rather different approach to the construction of blenders was introduced by Moreira and Silva [MS]. The basic idea is that, starting from a hyperbolic set whose fractal dimension is large enough to provide the desired additional dimensions, a blender will arise after a generic perturbation ("fractal transversality" argument). In their work, they succeeded in implementing this idea to obtain a blender yielding a single additional dimension.

Here we will show that if the dimension of the hyperbolic set is "very large", close to the dimension of the entire ambient manifold, then a superblender (a blender capable of yielding all desired additional dimensions), can be produced by a suitable perturbation involving a preliminary C^1 approximation of the hyperbolic set which is designed to trivialize the holonomies of the stable and unstable foliations restricted to a large subset. As it turns out, any regular perturbation of an ergodic nonuniformly Anosov map admits such very large hyperbolic sets. Using Theorem A', we can then conclude that superblenders appear C^1 densely among partially hyperbolic dynamical systems, and Theorem B follows.

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Discussion and questions. We return to the original question:

Question 1. *Is ergodicity a generic property in* $\mathrm{Diff}^1_m(M)$ *or in* $\mathrm{Diff}^2_m(M)$*?*

Some partial results are known. Bonatti and Crovisier proved [BC] that transitivity (i.e., existence of a dense orbit) is a generic property in $\mathrm{Diff}^1_m(M)$ (the topological mixing also holds [AC]). Recall that an important ingredient in our proof of Theorem B (which follows from accessibility) is metric transitivity, where almost every orbit is dense. A weaker problem than Question 1 is thus:

Question 2. *Is metric transitivity generic in* $\mathrm{Diff}^1_m(M)$*?*

The next question relates to the Oxtoby-Ulam technique. If f has entropy 0, then the results in [BDP], [BC] and [Av] show that it can be perturbed to have a dense set of periodic balls.

Question 3. *Can every* $f \in \mathrm{Diff}^1_m(M)$ *of entropy* 0 *be* C^1 *approximated by an almost everywhere periodic diffeomorphism (i.e. a diffeomorphism whose periodic points have full measure)?*

In the case of C^1-generic diffeomorphisms with positive entropy, a next goal would be to describe better their measurable dynamics. Some additional argument gives the following corollary of Theorem A:

Corollary A". *The generic $f \in \mathrm{Diff}^1_m(M)$ with positive metric entropy is weakly mixing.*

Corollary A" is proved in Section 4.2.

Due to the lack of regularity in $\mathrm{Diff}^1_m(M)$ we cannot use Pesin theory to get the Bernoulli property to be generic (as we do in the C^2 context of Theorem B).

Question 4. *Are generic non-uniformly Anosov diffeomorphisms in $\mathrm{Diff}^1_m(M)$ Bernoulli? or at least strongly mixing?*

As mentioned above, a $C^{1+\alpha}$-regularity hypothesis is needed for the Hopf argument and it is still unknown if stable ergodicity can happen in the C^1-topology. In particular:

Question 5. *Does there exist a non-ergodic volume preserving Anosov C^1-diffeomorphism on a connected manifold?*

For smoother systems, Tahzibi has shown [T2] that stable ergodicity can hold beyond the partially hyperbolic diffeomorphisms. One can thus hope to characterize stable ergodicity by the existence of a dominated splitting. The following conjecture could be compared to [DW, Conjecture 0.3] about robust transitivity.

Conjecture 6. *The sets of stably ergodic diffeomorphisms and of those having a non-trivial dominated splitting have the same C^1-closure in $\mathrm{Diff}^r_m(M)$, $r > 1$.*

If one considers the space $\mathrm{Diff}^r_\omega(M)$ of C^r-diffeomorphisms preserving a symplectic structure ω, the corresponding version of Theorem A has been established with different technics by Avila, Bochi and Wilkinson [ABW]: *The generic diffeomorphism in $\mathrm{Diff}^1_\omega(M)$ has either all Lyapunov exponents vanishing almost everywhere or the system is partially hyperbolic and ergodic.*

Note that in the second case the diffeomorphism is not always nonuniformly Anosov. For that reason we cannot build blenders and obtain a symplectic version of Theorem B.

Question 7. *Is stable ergodicity C^1-dense in the space $\mathrm{Diff}^r_\omega(M)$, $r > 1$?*

1. Further results and techniques in the proof

In the proofs of Theorems A and B, we introduce several techniques that might be of independent interest and which we present now.

We first fix some notations. Let M be a compact, connected boundaryless manifold with a fixed Riemannian metric, and denote by m the volume in this metric, normalized so that $m(M) = 1$. Sometimes we will also consider a symplectic structure ω and its associated normalized volume m. The spaces of C^r, volume preserving and symplectic diffeomorphisms are denoted by $\mathrm{Diff}^r_m(M)$ and $\mathrm{Diff}^r_\omega(M)$, respectively.

For $g \in \mathrm{Diff}^1_m(M)$, $x \in M$, and a subspace $F \subset T_xM$, we denote by $\mathrm{Jac}_F(g,x) > 0$ the Jacobian of Dg restricted to F, i.e., the product of the singular values of $Dg(x)|F$.

1.1. **Localized, pointwise perturbations of central Lyapunov exponents.** In order to prove the non-uniform hyperbolicity in Theorem A, we have to perturb the Lyapunov exponents on an invariant region. This is obtained through the following local, pointwise version of Bonatti-Baraviera's argument [BB] (see also Shub-Wilkinson [SW]). A more precise statement will be given in Section 3. If μ and ν are finite Borel measures on M, the notation $\mu \leq \nu$ means that $\mu(A) \leq \nu(A)$ for all measurable sets A.

Theorem C. *Let $f \in \mathrm{Diff}^1_m(M)$, and let $K \subset M$ be an invariant compact set such that:*

- *K admits a dominated splitting $T_K M = E_1 \oplus E_2 \oplus E_3$ into three non-trivial subbundles;*
- *for almost every point $x \in K$ one has*
$$\limsup_{n\to\pm\infty} \frac{1}{n} \log \mathrm{Jac}_{E_2(x)}(f^n, x) \leq 0.$$

Then for every $\varepsilon > 0$ and every small neighborhood Q of K, there exists a diffeomorphism g arbitrarily close to f in $\mathrm{Diff}^1_m(M)$ such that for every g-invariant measure ν such that $\nu \leq m|Q$ and $\nu(M) \geq \varepsilon$, one has $\int \log \mathrm{Jac}_{E_2(g,x)}(g, x) d\nu(x) < 0$.

In the previous statement $E_1(g) \oplus E_2(g) \oplus E_3(g)$ denotes the *continuation of the dominated splitting* for the diffeomorphism g on any g-invariant set contained in a neighborhood of Q. (See Section 2.1.2.)

Remark 1.1. Even for a diffeomorphism that preserves a globally partially hyperbolic structure, this is a new result. It produces changes in not just the average Lyapunov exponents, but the actual Lyapunov exponents. Without an assumption of ergodicity, these can be different.

1.2. **Linearization of horseshoes.** An essential tool for approximation in C^1 dynamics is a simple technique known as the "Franks' lemma". It asserts that for any periodic orbit $\mathcal{O}$ of a diffeomorphism f, there is is a C^1 small perturbation g of f, supported in a neighborhood of $\mathcal{O}$, such that g is affine near $\mathcal{O}$. Further perturbation is then vastly simplified starting from this affine setting. We introduce and prove here an analogue of the Franks' lemma for horseshoes.

Recall that a *horseshoe* for a diffeomorphism f is a transitive, locally maximal hyperbolic set Λ, that is totally disconnected and not finite (such a set must be perfect, hence a Cantor set).

Definition 1.2. A horseshoe Λ is *affine* if there exists a neighborhood U of Λ and a chart $\varphi\colon U \to \mathbb{R}^d$ such that $\varphi \circ f \circ \varphi^{-1}$ is locally affine near each point of Λ.

If one can choose φ such that the linear part of $D(\varphi f \varphi^{-1})(x)$ coincides with some $A \in \mathrm{GL}(d, \mathbb{R})$ independent of x, we say that Λ has *constant linear part A*.

The next result is proved in Section 6.

Theorem 1.3 (Linearization)**.** *Consider a C^r diffeomorphism f with $r \geq 1$, a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$ (in $\mathrm{Diff}^1_m(M)$ or in $\mathrm{Diff}^1_\omega(M)$ if f preserves the volume m or the symplectic form ω), a horseshoe Λ and $\varepsilon > 0$. Then there exist a C^r diffeomorphism $g \in \mathcal{U}$ with an affine horseshoe $\widetilde{\Lambda}$ such that:*

- *$g = f$ outside the ε-neighborhood of Λ.*

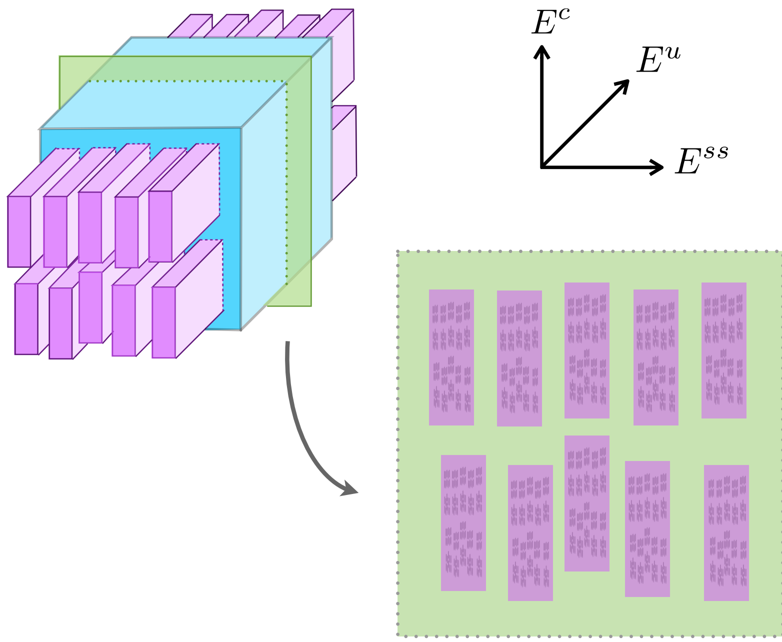

FIGURE 1. An affine horseshoe with constant linear part and its slice inside a stable manifold.

– *$\widetilde{\Lambda}$ is ε-close to Λ in the Hausdorff distance.*
– *$h_{top}(\widetilde{\Lambda}, g) > h_{top}(\Lambda, f) - \varepsilon$.*

Moreover there exists a linearizing chart $\varphi\colon U \subset M \to \mathbb{R}^d$ with $\widetilde{\Lambda} \subset U$ and a diagonal matrix A with diagonal coefficients that are all distinct such that f coincides with the affine map $z \mapsto A.(z-x) + f(x)$ in a neighborhood of each point $x \in \widetilde{\Lambda}$.

Such a result is false for stronger topologies. However, one can "diagonalize" a sub horseshoe, as follows. This result is proved in Section 5, and is used in the proof of Theorem 1.3.

Theorem 1.4 (Diagonalization)**.** *Consider a C^k diffeomorphism f with $k \geq 1$, a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^k(M)$ (in $\mathrm{Diff}^k_m(M)$ or in $\mathrm{Diff}^k_\omega(M)$ if f preserves the volume m or the symplectic form ω), a horseshoe Λ and $\varepsilon > 0$. Then there exist a C^k diffeomorphism $g \in \mathcal{U}$ with a horseshoe $\widetilde{\Lambda}$ such that:*

– *$g = f$ outside the ε-neighborhood of Λ.*
– *$\widetilde{\Lambda}$ is ε-close to Λ in the Hausdorff distance.*
– *$h_{top}(\widetilde{\Lambda}, g) > h_{top}(\Lambda, f) - \varepsilon$.*
– *Λ_g admits a dominated splitting into one-dimensional sub bundles*

$$T_\Lambda = E_1 \oplus \cdots \oplus E_d.$$

1.3. **Approximation of hyperbolic measures by affine horseshoes.** For C^r diffeomorphisms, a theorem by Katok [Ka] asserts that any hyperbolic ergodic measure can be approximated by a horseshoe. It is possible to do this so that the horseshoe has a dominated splitting ,with approximately the same Lyapunov exponents on the horseshoe. Strictly speaking, the original result of Katok does not explicitly mention such a control of the Oseledets splitting, but no further work is really needed to obtain it. Since we have not been able to find out this precise statement in the litterature, we include a proof of the following version of Katok's theorem in Section 8.

Theorem 1.5 (Katok's approximation)**.** *Consider $r > 1$, a C^r-diffeomorphism f, an ergodic, f-invariant, hyperbolic probability measure μ, a constant $\delta > 0$, and a weak-* neighborhood $\mathcal{V}$ of μ in the space of f-invariant probability measures on M. Then there exists a horseshoe $\Lambda \subset M$ such that:*

1. *Λ is δ-close to the support of μ in the Hausdorff distance;*
2. *$h_{top}(\Lambda, f) > h(\mu, f) - \delta$;*
3. *all the invariant probability measures supported on Λ lie in $\mathcal{V}$;*
4. *if $\chi_1 > \cdots > \chi_\ell$ are the distinct Lyapunov exponents of μ, with multiplicities $n_1, \dots, n_\ell \geq 1$, then there exists a dominated splitting on Λ:*
$$T_\Lambda M = E_1 \oplus \cdots \oplus E_\ell, \qquad \textit{with } \dim(E_i) = n_i;$$
5. *there exists $n \geq 1$ such that for each $i = 1, \dots, \ell$, each $x \in \Lambda$ and each unit vector $v \in E_i(x)$,*
$$\exp((\chi_i - \delta)n) \leq \|Df_0^n(v)\| \leq \exp((\chi_i + \delta)n).$$

This can be combined with Theorem 1.4 in order to obtain (after a C^1-perturbation) an affine horseshoe that approximates the measure.

Theorem D. *Consider $r > 1$, a C^r diffeomorphism f, a C^1-neighborhood $\mathcal{U} \subset \mathrm{Diff}^r(M)$ of f, a hyperbolic ergodic probability measure μ, a constant $\delta > 0$ and a weak-* neighborhood $\mathcal{V}$ of μ in the space of f-invariant probability measures on M. There exists $g \in \mathcal{U}$ and an affine horseshoe Λ with constant linear part A such that:*

- *Λ is δ-close to the support of μ in the Hausdorff distance;*
- *$h_{top}(\Lambda, g) > h(\mu, f) - \delta$;*
- *all the g-invariant probability measures supported on Λ lie in $\mathcal{V}$;*
- *A is diagonal, with distinct real positive eigenvalues whose logarithms $\lambda_1 > \cdots > \lambda_d$ are δ-close to the Lyapunov exponents of μ (with multiplicity).*

If f preserves the volume m or a symplectic form ω, then g can be chosen to preserve it as well.

1.4. **Blenders.** The data for a stable blender are: a horseshoe Λ with a partially hyperbolic subsplitting $T_\Lambda M = E^{uu} \oplus E^c \oplus E^s$, the blender itself, which is a local chart ("box") centered at a point of the horseshoe, and finally a conefield in the box that contains E^{uu} at points of Λ. The blender property requires that any disk tangent to the E^{uu} cone and crossing the box meets the stable manifold of Λ (see below for a formal definition).

The dimension of the center bundle of the splitting in some sense describes the strength of the blender. In the classical blender construction, the dimension of E^c is low – either 1 or 2 [BD, RRTU]. The reason for the low-dimensionality of E^c in these constructions is the challenge of controlling the dynamics of f in the central direction. Roughly, the smaller the dimension of E^c, the less wiggle room for a uu-disk to avoid the stable manifold of Λ. The other extreme, where $\dim(E^{uu}) = 1$ and $\dim(E^c)$ is arbitrary, is a "superblender," which is what we construct here.

In order to give a precise definition, we fix an integer $1 \leq d_{cs} \leq d := \dim(M)$.

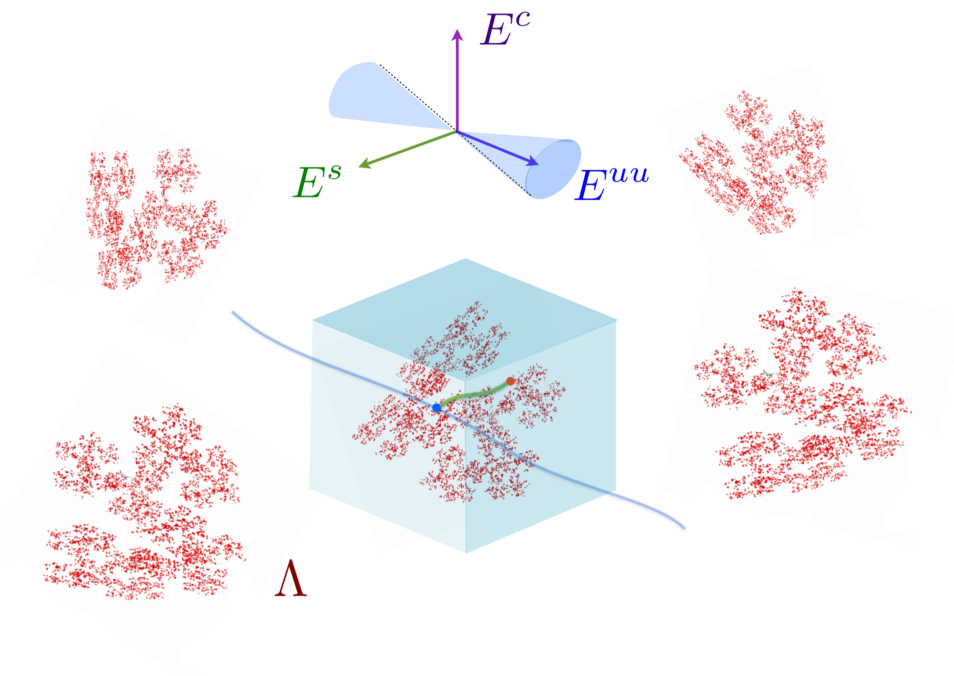

FIGURE 2. A stable blender

Definition 1.6. A horseshoe Λ with a dominated splitting

$$T_\Lambda M = E^u \oplus E^s = (E^{uu} \oplus E^c) \oplus E^s, \quad \dim(E^c \oplus E^s) = d_{cs}$$

is a d_{cs}-*stable blender* if there is a chart $\varphi\colon U \to (-1,1)^d$ of M such that

- $x = \varphi^{-1}(0)$ belongs to Λ and $\varphi^{-1}((-1,1)^{d-d_{cs}} \times \{0\}^{d_{cs}})$ is contained in the strong unstable manifold of x tangent to $E^{uu}(x)$,
- the graph $\{(x, \theta(x))\}$ of any 1-Lipschitz map $\theta\colon (-1,1)^{d-d_{cs}} \to (-1,1)^{d_{cs}}$ meets the local stable set of the hyperbolic continuation Λ_g of Λ for each diffeomorphism g that is C^1-close to f.

This definition asserts that the stable set of Λ behaves as a d_{cs}-dimensional space transverse to the strong unstable direction. We define similarly the notion of d_{cu}-*unstable blender*. We say that Λ is a *stable superblender* if it is a k-stable blender for all $k \in [\dim(E^s), d-1]$. Equivalently, Λ is a stable superblender if it is a $(d-1)$-stable blender and moreover its unstable bundle splits as dominated sum of one-dimensional subbundles:

$$E^u = E^u_1 \oplus \cdots \oplus E^u_{d-d_s}.$$

Analogously, Λ is an *unstable superblender* if it is a k-unstable blender for all $k \in [d_u, d-1]$. Finally, Λ is a *superblender* if it is both a stable and unstable superblender.

Blenders are obtained with the following theorem, proved in Section 7.

Theorem E. *Consider an integer $k \geq 1$, a C^k-diffeomorphism f and an affine horseshoe Λ of f with constant linear part $A \in \mathrm{GL}(d, \mathbb{R})$ such that:*

- *A preserves a dominated decomposition $\mathbb{R}^d = E^u \oplus E^s = (E^{uu} \oplus E^c) \oplus E^s$.*
- *A^{-1} is a contraction on $E^{uu} \oplus E^c$ and A is a contraction on E^s.*
- *The measure of maximal entropy on Λ "almost satisfies" the Pesin formula:*

$$h_{top}(\Lambda, f) > \log \mathrm{Jac}_{E^u}(A) - \frac{1}{2k}\chi^u_{inf}(A), \tag{5}$$

where $\chi^u_{inf}(A)$ is the smallest positive Lyapunov exponent of A.

Then there exists a C^k-perturbation g of f supported in a small neighborhood of Λ such that the hyperbolic continuation Λ_g is a d_{cs}-stable blender. If f preserves the volume m, then one can choose g to preserve it also.

We elaborate on the final hypothesis of Theorem E. In [P] Pesin proved that the Ruelle inequality (3) becomes equality in the case where f is C^2 and the invariant measure μ is the volume m:

$$h_m(f) = \int_M \sum_{\chi_i(x) \geq 0} \dim(E_i(x))\chi_i(x)\, dm(x). \tag{6}$$

More generally, equality (6) holds precisely when the invariant measure m has absolutely continuous disintegration along Pesin unstable manifolds [LY1]. In particular, if m is supported on a (proper) C^2 horseshoe, (6) will never hold. We can nonetheless quantify how close m comes to satisfying (6); the final hypothesis of Theorem E requires that the measure of maximal entropy for the horseshoe Λ (whose entropy is equal to $h_{top}(\Lambda, f)$) be "fat" along unstable manifolds. Since for m-a.e. point x the sum of the positive Lyapunov exponents $\chi_i(x)$ counted with their multiplicity $\dim(E_i(x))$ coincides with $\log \mathrm{Jac}_{E^u}(A)$, the equality (6) almost holds.

The entropy and the positive Lyapunov exponents are related to unstable dimensions d_i through the Ledrappier-Young formula [LY2]:

$$h_{top}(f) = \sum_{\chi_i(x) \geq 0} d_i \chi_i(x).$$

Condition (5) implies that the sum of unstable dimensions d_i is larger than $d^u - \frac{1}{2k}$. In the case $d^c = 1$, Moreira and Silva have obtained [MS] a much stronger result, valid in the C^∞-topology, even for non-affine horseshoe, assuming that the "upper-unstable dimension" of Λ is larger than 1. Perturbations tend to increase the dimensions associated to the lower Lyapunov exponents and to decrease the others. Consequently one can expect that an optimal hypothesis in Theorem E would be:

$$h_{top}(\Lambda, f) > \log \mathrm{Jac}_{E^c}(A).$$

Denote by $\mathcal{DS}$ the set of diffeomorphisms in $\mathrm{Diff}^1_m(M)$ with a nontrivial dominated splitting. From Theorems A, D, and E, we obtain (see Section 4):

Theorem F. *Any diffeomorphism f in a dense set of $\mathcal{DS}$ has a superblender Λ.*

Moreover there exists a dominated splitting $TM = E \oplus F$ such that $\dim(E)$ coincides with the unstable dimension of Λ and, for any diffeomorphism C^1-close to f, the set of points having $\dim(E)$ positive Lyapunov exponents and $\dim(F)$ negative Lyapunov exponents has positive volume.

Theorem B is proved in Section 4 by combining Theorem F and the criterion for ergodicity obtained in [RRTU].

2. A dichotomy for conservative diffeomorphisms

In this section we prove Theorems A and A' assuming Theorem C.

2.1. **Dominated splittings and center Jacobians.** Let $f \in \mathrm{Diff}^1(M)$. We recall well-known properties of dominated splittings.

2.1.1. Given an f-invariant compact set K_f, we say that $f|K_f$ admits a *dominated splitting of type* (d_1, d_2, d_3) (where $d_1, d_2, d_3 \geq 0$ and $d_1 + d_2 + d_3 = d$) if there is an f-invariant splitting $T_xM = E_1(x) \oplus E_2(x) \oplus E_3(x)$ defined over K, where $E_*(x) = E_*(f, x)$ are subspaces of dimension d_*, $* = 1, 2, 3$, and there is an $n \in \mathbb{N}$ such that for each $x \in K$ one has

$$\|(Df^n(x)|E_1(x))^{-1}\| < \|Df^n(x)|E_2(x)\|^{-1},$$
$$\|(Df^n(x)|E_2(x))^{-1}\| < \|Df^n(x)|E_3(x)\|^{-1}.$$

In other words, the smallest contraction along E_1 and the largest expansion along E_3 dominate the behavior along E_2. The $E_*(x)$ are uniquely defined in this way and depend continuously on x.

2.1.2. This dominated splitting is robust in the following sense. Consider an arbitrary continuous extension of the $E_*(x)$ to a neighborhood of K_f and consider arbitrary metrics on the Grassmanian manifolds of M. Then for every $\alpha > 0$, there are neighborhoods $\mathcal{V} \subset \mathrm{Diff}^1(M)$ of f and $V \subset M$ of K_f such that if $g \in \mathcal{V}$ and $K_g \subset V$ is a compact invariant set, then $g|K_g$ admits a dominated splitting of type (d_1, d_2, d_3), and moreover the spaces $E_*(g, x)$ are α-close to (the extension of) $E_*(f, x)$ for every $x \in K_g$.

2.1.3. Given a compact set $Q \subset M$, we let $K(f, Q) = \bigcap_{n \in \mathbb{Z}} f^n(Q)$ be its maximal f-invariant subset. Notice that $K(g, Q) \subset V$ for every neighborhood V of $K(f, Q)$ and every $g \in \mathrm{Diff}^1(M)$ close to f in the C^0 topology.

The previous paragraph thus implies that *the set of all* $g \in \mathrm{Diff}^1(M)$ *such that* $g|K(g, Q)$ *admits a dominated splitting of type* (d_1, d_2, d_3) *is open.*

2.1.4. Let $X_f \subset M$ be the set of *Oseledets regular points* x of f, i.e. which have well-defined Oseledets splitting and Lyapunov exponents

$$\lambda_1(f, x) \geq \lambda_2(f, x) \geq \cdots \geq \lambda_d(f, x).$$

By Oseledets's theorem, X_f is a measurable f-invariant set with full measure for any f-invariant Borel probability measure μ. Moreover, the Lyapunov exponents define d functions $\lambda_1, \dots, \lambda_d \in L^1(\mu)$.

For any regular point x, by summing all the directions associated to the positive, zero, or negative Lyapunov exponents, we obtain a splitting:

$$T_xM = E^+(x) \oplus E^0(x) \oplus E^-(x).$$

The Pesin stable manifold theorem asserts that for $\varepsilon > 0$ small,

$$W^-(x) := \{z : \limsup_{n \to +\infty} \frac{1}{n} \log d(f^n(x), f^n(z)) \leq -\varepsilon\}$$

is an injectively immersed submanifold tangent to $E^-(x)$. Symmetrically, one obtains an injectively immersed submanifold $W^+(x)$ tangent to $E^+(x)$. The dimensions $\dim(E^+(x))$, $\dim(E^-(x))$ are called *unstable* and *stable dimensions* of x.

An invariant probability measure is *hyperbolic* if for almost every point the Lyapunov exponents are all different from zero.

2.1.5. For $x \in M$ and a subspace $F \subset T_xM$, we let

$$\Delta_F(f,x) = \limsup_{n\to\pm\infty} \frac{1}{n} \log \mathrm{Jac}_F(f^n,x).$$

If x is Oseledets regular, and F is a sum of Oseledets subspaces, then $\frac{1}{n}\log \mathrm{Jac}_F(f^n,x)$ indeed converges to the sum of the Lyapunov exponents of f along F. Moreover if ν is an f-invariant finite Borel measure, and $F(x) \subset T_xM$ is measurable f-invariant distribution of subspaces defined ν-almost everywhere, then for every $n \geq 1$ we have

$$\int \Delta_{F(x)}(f,x)d\nu(x) = \frac{1}{n}\int \log \mathrm{Jac}_{F(x)}(f^n,x)d\nu(x).$$

2.1.6. If μ and ν are finite Borel measures, the notation $\mu \leq \nu$ means that $\mu(A) \leq \nu(A)$ for all measurable sets A. This property is equivalent to the two conditions: μ is absolutely continuous with respect to ν and the Radon-Nikodym derivative $d\mu/d\nu$ is essentially bounded above by 1. When ν is fixed, the set of measures μ satisfying $\mu \leq \nu$ is clearly compact in the weak-$*$ topology.

2.1.7. Recall that m is a smooth volume on M. For $\varepsilon > 0$ and $Q \subset M$ compact, we denote by $\mathcal{G}_\varepsilon(Q,d_1,d_2,d_3)$ the set of all $g \in \mathrm{Diff}^1(M)$ such that

- $g|K(g,Q)$ admits a dominated splitting of type (d_1,d_2,d_3),
- for every g-invariant measure $\nu \leq m|Q$ satisfying $\nu(M) \geq \varepsilon$, one has

$$\int \mathrm{Jac}_{E_2(g,x)}(g,x)d\nu(x) < 0.$$

Lemma 2.1. *For every $\varepsilon > 0$, the set $\mathcal{G}_\varepsilon(Q,d_1,d_2,d_3)$ is open in* $\mathrm{Diff}^1(M)$.

Proof. Consider (g_n) converging to g in $\mathrm{Diff}^1(M)$ and assume $g_n \notin \mathcal{G}_\varepsilon(Q,d_1,d_2,d_3)$. We have to prove that $g \notin \mathcal{G}_\varepsilon(Q,d_1,d_2,d_3)$. For the sake of contradiction, by Section 2.1.2 it suffices to assume that the g_n admit dominated splittings of type (d_1,d_2,d_3). Let $\nu_n \leq m|Q$ be a sequence of g_n-invariant measures satisfying $\nu_n(M) \geq \varepsilon$ and $\int \mathrm{Jac}_{E_2(g_n,x)}(g_n,x)d\nu_n(x) \geq 0$. Let ν be a weak-$*$ limit of ν_n. Then $\nu \leq m|Q$ is g-invariant and satisfies $\nu(M) \geq \varepsilon$ and $\int \mathrm{Jac}_{E_2(g,x)}\, d\nu(x) \geq 0$. Hence $g \notin \mathcal{G}_\varepsilon(Q,d_1,d_2,d_3)$. □

2.2. **Oseledets blocks.** For $f \in \mathrm{Diff}^1_m(M)$, the set of regular points X_f splits into f-invariant measurable subsets $X_f(d_1,d_2,d_3)$, $d_1+d_2+d_3 = d$ and $d_* \geq 0$, defined as the set of points admitting d_1 positive, d_2 zero and and d_3 negative Lyapunov exponents (counted with multiplicity). Note that:

- $X_f(0,d,0)$ is the set of points whose Lyapunov exponents are all zero;
- the set of *non-uniformly hyperbolic points*, denoted by Nuh_f is the union of the sets $X(d_1,0,d_3)$, with $d_1,d_3 > 0$;
- by volume preservation, the other non-empty sets satisfy $d_1,d_2,d_3 > 0$.

2.2.1. *Domination.* Oseledets and dominated splittings coincide generically.

Theorem 2.2 (Bochi-Viana [BV2]). *For any diffeomorphism f in a dense G_δ subset of $\mathrm{Diff}^1_m(M)$ and for any $\varepsilon > 0$, for each Oseledets block $X_f(d_1, d_2, d_3)$ there exists an f-invariant compact set K satisfying:*

- *$f|K$ admits a dominated splitting of type (d_1, d_2, d_3),*
- *$m(X_f(d_1, d_2, d_3) \setminus K) \leq \varepsilon$.*

2.2.2. *Continuation.* Generically the Oseledets blocks vary continuously.

Theorem 2.3 (Avila-Bochi [AB], Theorem D). *For each $i \in \{1, \dots, d\}$, the continuity points of the Lyapunov exponent functions $\lambda_i : \mathrm{Diff}^1_m(M) \to L^1(m)$ is a dense G_δ subset of $\mathrm{Diff}^1_m(M)$.*

Corollary 2.4. *For any diffeomorphism f in a dense G_δ set of $\mathrm{Diff}^1_m(M)$ and for any $\varepsilon > 0$, there exists a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1_m(M)$ such that each $g \in \mathcal{U}$ satisfies:*

$$m(\mathrm{Nuh}_f \setminus \mathrm{Nuh}_g) \leq \varepsilon.$$

2.2.3. *The non-uniformly hyperbolic set.* Generically the non-uniformly hyperbolic set Nuh_f coincides m-almost everywhere with a single Oseledets block.

Theorem 2.5 (Avila-Bochi [AB], Theorem A). *For any diffeomorphism f in a dense G_δ subset of $\mathrm{Diff}^1_m(M)$, either $m(\mathrm{Nuh}_f) = 0$ or Nuh_f is dense in M and the restriction $m|\,\mathrm{Nuh}_f$ is ergodic.*

2.2.4. *The set where all exponents vanish.* As a consequence we get (see also [AB], Corollary 1.1):

Corollary 2.6. *For any diffeomorphism f in a dense G_δ subset of $\mathrm{Diff}^1_m(M)$, such that $m(\mathrm{Nuh}_f) > 0$, there exists a global dominated splitting $TM = E \oplus F$ on M such that for m-almost every point $x \in \mathrm{Nuh}_f$*

$$v \in E(x) \setminus \{0\} \implies \lim_{n\to\infty} \frac{1}{n} \log \|D_x f^n(v)\| > 0,$$

$$v \in F(x) \setminus \{0\} \implies \lim_{n\to\infty} \frac{1}{n} \log \|D_x f^n(v)\| < 0.$$

In particular $X_f(0, d, 0) = \emptyset$.

Proof. By Theorem 2.5 if Nuh_f has positive volume, then it is dense in M, the restriction of m is ergodic and it coincides with a set $X(d_1, 0, d_3)$. By Theorem 2.2, there exists an invariant compact set K which admits a non-trivial dominated splitting. The measure $\mathrm{Nuh}_f \setminus K$ is arbitrarily small, hence is zero by ergodicity. This proves that the compact set K contains m-almost every point of Nuh_f, hence coincides with M. We have proved that M has a non-trivial dominated splitting, hence the set $X_f(0, d, 0)$ is empty. $\square$

2.2.5. *The other Oseledets blocks.* Using Theorem C we get:

Corollary 2.7. *For any diffeomorphism f in a dense G_δ subset of $\mathrm{Diff}^1_m(M)$, the Oseledets blocks $X_f(d_1, d_2, d_3)$ with $d_1, d_2, d_3 > 0$ have volume zero.*

Proof. Let $\mathcal{K}$ be a countable family of compact sets of M such that for any $K \subset U \subset M$, with K compact and U open, there exists $Q \in \mathcal{K}$ satisfying $K \subset Q \subset U$. By Lemma 2.1, one can assume that for any $Q \in \mathcal{K}$, any $\varepsilon > 0$ such that $1/\varepsilon \in \mathbb{N}$, and any type (d_1, d_2, d_3), the diffeomorphism f either belongs to $\mathcal{G}_\varepsilon(Q, d_1, d_2, d_3)$ or to $\mathrm{Diff}^1_m(M) \setminus \overline{\mathcal{G}_\varepsilon(Q, d_1, d_2, d_3)}$.

Assume first that Nuh_f has zero volume. We then prove by increasing induction on $d_2 + d_3$ in $\{1, \dots, d-1\}$ that $m(X_f(d_1, d_2, d_3)) = 0$, for each triple (d_1, d_2, d_3) with $d_1 + d_2 + d_3 = d$ and $d_* > 0$. We thus fix (d_1, d_2, d_3) and assume that $m(X_f(d'_1, d'_2, d'_3)) = 0$ for each triple (d'_1, d'_2, d'_3) such that $d'_* > 0$ and $d'_2 + d'_3 < d_2 + d_3$.

We fix $\varepsilon > 0$ with $1/\varepsilon \in \mathbb{N}$. By Theorem 2.2 there exists an invariant compact set K such that $m(X_f(d_1, d_2, d_3) \setminus K)$ is smaller than ε and such that $f|K$ admits a dominated splitting of type (d_1, d_2, d_3).

Almost every point $x \in K$ belongs to a set $X_f(d'_1, d'_2, d'_3)$ with positive volume. The possible values of d'_* are restricted; in particular:

- We have $X_f(d'_1, d'_2, d'_3) \neq X_f(0, d, 0)$, since $X_f(d'_1, d'_2, d'_3)$ intersects K, which has a non-trivial dominated splitting.
- We have $d_2 > 0$ since Nuh_f has zero volume.
- If d'_1, d'_2, d'_3 are all positive, our inductive assumption implies that $d'_2 + d'_3 \geq d_1 + d_2$. As a consequence $E_2(f, x)$ is contained in the sum of the central and the stable spaces of the Oseledets decomposition for f at x. This implies $\Delta_{E_2(f,x)}(f, x) \leq 0$.

We have proved that the assumptions of Theorem C are satisfied. We choose a small neighborhood $Q \in \mathcal{K}$ of K. There exists g arbitrarily close to f in $\mathrm{Diff}^1_m(M)$ such that for every invariant measure $\nu \leq m|Q$ such that $\nu(M) \geq \varepsilon$, one has $\int_X \log \mathrm{Jac}_{E_2(g,x)}\, d\nu(x) < 0$. In particular g belongs to $\mathcal{G}_\varepsilon(Q, d_1, d_2, d_3)$, and hence f does as well (recall that f belongs to the union of the open set $\mathcal{G}_\varepsilon(Q, d_1, d_2, d_3)$ with $\mathrm{Diff}^1_m(M) \setminus \overline{\mathcal{G}_\varepsilon(Q, d_1, d_2, d_3)}$). It follows that $X_f(d_1, d_2, d_3) \cap K$ has volume smaller than ε. With our choice of K, this proves $m(X_f(d_1, d_2, d_3)) \leq 2\varepsilon$. Since $\varepsilon > 0$ has been arbitrarily chosen we get $m(X_f(d_1, d_2, d_3)) = 0$, as desired.

In the case Nuh_f has positive volume, we modify the previous argument. By Theorem 2.5, there exists d_+, d_- such that Nuh_f and $X_f(d_+, 0, d_-)$ coincide up to a set of volume zero and by Corollary 2.6 there exists a global domination $TM = E \oplus F$ with $\dim(E) = d_+$. The only sets $X(d_1, d_2, d_3)$ that can have positive volume thus satisfy $d_2 + d_3 \leq d_-$ or $d_1 + d_2 \leq d_+$. We only deal with the first case, since the second is similar. We thus have to prove by increasing induction on $d_2 + d_3$ in $\{1, \dots, d_-\}$ that $m(X_f(d_1, d_2, d_3)) = 0$ for each triple (d_1, d_2, d_3) with $d_1 + d_2 + d_3 = d$ and $d_* > 0$. We repeat the same argument as before. This time the set K can intersect Nuh_f, but since $d_2 + d_3 \leq d_-$ we have $\Delta_{E_2(f,x)}(f, x) < 0$ for points $x \in K \cap \mathrm{Nuh}_f$. The end of the argument is unchanged. This completes the proof.

□

2.3. **Proof of Theorems A and A'.** The Theorem 2.5 and the Corollaries 2.6 and 2.7 now imply Theorem A'. Theorem A follows immediately as explained in the introduction.

3. Local perturbations of center exponents

This section is devoted to the proof of the following, which implies Theorem C.

Theorem C'. *Let $f \in \mathrm{Diff}^1_m(M)$ and let K be an f-invariant compact set admitting a dominated splitting $T_K M = E_1 \oplus E_2 \oplus E_3$ into three non-trivial subbundles. Then for any $\alpha > 0$ small and for any neighborhood $\mathcal{U} \subset \mathrm{Diff}^1_m(M)$ of the identity, there exist $\delta > 0$ and $n_0 \geq 1$ with the following property.*

For any $n \geq n_0$, any neighborhood Q of K and any $\eta > 0$, there exist a smooth diffeomorphism $\varphi \in \mathcal{U}$, and a measurable subset $\Lambda \subset Q$ such that:

- *φ is supported on Q and is η-close to the identity in the C^0 topology,*
- *$m(K \setminus \Lambda) < \eta$,*
- *the diffeomorphism $g = f \circ \varphi$ satisfies*

$$\frac{1}{n}\log \mathrm{Jac}_F(g^n, y) \leq \frac{1}{n}\log \mathrm{Jac}_{E_2(f,y)}(f^n, y) - \delta, \tag{7}$$

for every $y \in \Lambda$ such that $y, g^n(y) \in K$, and every subspace $F \subset T_y M$ such that F is α-close to $E_2(f,y)$ and $Dg^n(y)\cdot F$ is α-close to $E_2(f, g^n(y))$.

Proof of Theorem C from Theorem C'. Consider f, K, ε as in the statement of Theorem C and small neighborhoods $\mathcal{U} \subset \mathrm{Diff}^1_m(M)$ of f and $Q \subset M$ of K such that the maximal invariant set $K(g,Q)$ for any $g \in \mathcal{U}$ still has a dominated splitting which extends the splitting $T_K M = E_1 \oplus E_2 \oplus E_3$ on K. We have to find g satisfying the conclusion of the Theorem C.

Fix $\alpha > 0$ small. Reducing $\mathcal{U}, Q$ if necessary, for any point $x \in K(g,Q) \cap K$ the spaces $E_2(x,f)$ and $E_2(g,x)$ are α-close. Theorem C' applied to $\alpha, \mathcal{U}$ gives δ, n_0. Let C_0 be an upper bound for $d \log \|Dg(x)\|$, where $x \in M$, $g \in \mathcal{U}$. We also take $\kappa > 0$ smaller than $\min(\varepsilon/6, \delta\varepsilon/12C_0)$.

We choose $n \geq n_0$ and define the set

$$\Omega = \{x \in K,\ \frac{1}{n}\log \mathrm{Jac}_{E^0(f,x)}(f^n, x) \leq \delta/2\}.$$

If n is large enough, $K \setminus \Omega$ has measure less than κ. For $\eta > 0$ sufficiently small, shrinking if necessary the neighborhood Q, for any g such that $g \circ f^{-1}$ is η-close to the identity in the C^0 topology, we have:

$$m(K \setminus g^{-n}(K)) \leq \kappa, \quad m(K(g,Q) \setminus K) \leq \kappa.$$

Theorem C' provides us with a diffeomorphism g and a set Λ such that for every $x \in K(g,Q) \cap K \cap \Lambda \cap \Omega \cap g^{-n}(K)$ one has

$$\frac{1}{n}\log \mathrm{Jac}_{E_2(g,x)}(g^n, x) \leq \frac{1}{n}\log \mathrm{Jac}_{E_2(f,x)}(f^n, x) - \delta \leq \delta/2.$$

Moreover the complement of the set $Z := K(g,Q) \cap K \cap \Lambda \cap \Omega \cap g^{-n}(K)$ in $K(g,Q)$ has volume smaller than 3κ.

If $\nu \leq m|Q$ is a g-invariant measure with $\nu(M) \geq \varepsilon$, then $\nu(Z) \geq \varepsilon - 3\kappa \geq \varepsilon/2$. Thus

$$\begin{aligned}\int \log \mathrm{Jac}_{E_2(g,x)}(g,x)\,d\nu(x) &= \int \frac{1}{n}\log \mathrm{Jac}_{E_2(g,x)}(g^n,x)\,d\nu(x)\\ &\leq C_0\nu(M\setminus Z) - \frac{\delta}{2}\nu(Z) < 3C_0\kappa - \frac{\delta\varepsilon}{4} < 0.\end{aligned}$$

The result follows. $\square$

The construction of the perturbation in Theorem C' follows three natural steps, and will occupy the remainder of this section.

3.1. **Infinitesimal.** Let $\mathbb{R}^d = E^+ \oplus E^0 \oplus E^-$ be an orthogonal decomposition, and set $d_0 = \dim(E^0)$. Let $G \subset \mathbb{R}^d$ be a two-dimensional subspace that intersects both E^0 and E^- in one-dimensional subspaces, endowed with an arbitrary orientation. For a subspace $F \subset \mathbb{R}^d$, we let $F^\perp$ denote its orthogonal complement, and we let $P_F : \mathbb{R}^d \to F$ be the projection with kernel $F^\perp$. For $\theta \in \mathbb{R}$, let $R_\theta : \mathbb{R}^d \to \mathbb{R}^d$ be the orthogonal operator that is the identity on $G^\perp$, and that restricted to G is a rotation of angle $2\pi\theta$ (measured according to the chosen orientation).

Elementary perturbation. We introduce a diffeomorphism ψ^ε which will be used at different places for the perturbation. Let $\alpha : \mathbb{R}^d \to \mathbb{R}$ be a smooth function with the following properties:

- $\alpha(x) = 0$ for x in the complement of the unit ball $B := \{x, \|x\| \geq 1\}$,
- $\alpha(x) = 1$ for $\|x\| \leq 1/2$,
- $\|\alpha\|_{C^0} \leq 1$,
- $\alpha(R_\theta \cdot x) = \alpha(x)$ for every $\theta \in \mathbb{R}$ and $x \in \mathbb{R}^d$.

Given $\varepsilon > 0$, let $\psi^\varepsilon : \mathbb{R}^d \to \mathbb{R}^d$ be defined by $\psi^\varepsilon(x) = R_{\varepsilon\alpha(x)} \cdot x$. It is a smooth, volume preserving diffeomorphism of $\mathbb{R}^d$ and is the identity outside the unit ball. See Figure 3. For every $k \in \mathbb{N}$, we have $\|\psi^\varepsilon\|_{C^k} \leq \kappa(k)\varepsilon$ for some constant $\kappa(k) > 0$.

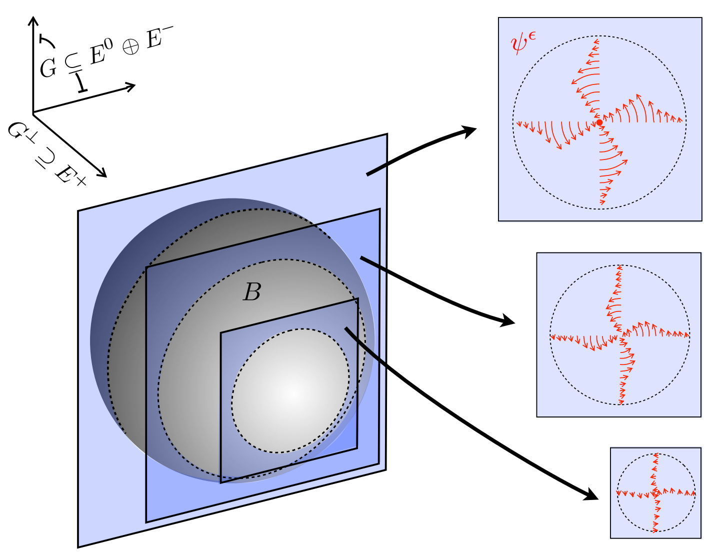

FIGURE 3. The map ψ^ε.

Let μ_ε be a probability measure in $\mathrm{SL}(d, \mathbb{R})$ given by the push-forward under $x \mapsto D\psi^\varepsilon(x)$ of normalized Lebesgue measure m on the unit ball. Note that for every $A \in \operatorname{supp} \mu_\varepsilon$, we have $A \cdot (E^0 + G) = (E^0 + G)$. We set

$$c(\varepsilon) = -\int \log \operatorname{Jac}_{E^0}(P_{E^0} \cdot A) d\mu_\varepsilon(A).$$

We describe the effect of an elementary perturbation averaged on the unit ball.

Lemma 3.1. *For every $\varepsilon > 0$ sufficiently small, we have $c(\varepsilon) > 0$.*

Proof. Notice that for any $x_0 \in G^\perp$, $x \mapsto P_{E^0} \cdot \psi^\varepsilon(x_0 + x)$ defines a diffeomorphism of F which is the identity outside the ball of radius $(1 - |x_0|^2)^{1/2}$. In particular,

$$\int \mathrm{Jac}_{E^0}(P_{E^0} \cdot A) d\mu_\varepsilon(A) = \int \mathrm{Jac}_{E^0}(P_{E^0} \cdot D\psi^\varepsilon(x)) dm(x) = 1.$$

Observe also that for $|x| < 1/2$ we have $\mathrm{Jac}_{E^0}(P_{E^0} \cdot D\psi^\varepsilon(x)) = \cos(2\pi\varepsilon) < 1$. Thus $c(\varepsilon) > 0$ follows from Jensen's inequality. $\square$

Random composition of elementary perturbations. By the Law of Large Numbers, the effect of an elementary perturbation composed along most random sequences of points of the unit ball is the same as the average effect of a single elementary perturbation.

Proposition 3.2. *If $\varepsilon > 0$ is small, there exists $\lambda \in (0, 1/4)$ such that for every $\theta > 0$ there exists $R_0 \in \mathbb{N}$ with the following property. Let $R \geq R_0$ and let $L_j : \mathbb{R}^d \to \mathbb{R}^d$, $0 \leq j \leq R-1$, be invertible linear operators preserving E^+, E^0 and E^- such that*

$$\|L_j|E^0\| \cdot \|L_j^{-1}|E^+\| \leq \lambda \quad \text{and} \quad \|L_j|E^-\| \cdot \|L_j^{-1}|E^0\| \leq \lambda.$$

Then there exists a compact set $W \subset \mathrm{SL}(d, \mathbb{R})^R$ with $\mu_\varepsilon^{\otimes R}(\mathrm{SL}(d, \mathbb{R})^R \setminus W) < \theta$ such that

$$\log \mathrm{Jac}_F \big((L_{R-1} \cdot A_{R-1}) \cdots (L_1 \cdot A_1) \cdot (L_0 \cdot A_0)\big) < \sum_{j=0}^{R-1} \log \mathrm{Jac}_{E^0}(L_j) - \frac{c(\varepsilon)}{2} R,$$

for every $(A_0, \dots A_{R_1}) \in W$ and for every d_0-dimensional subspace F such that $\|P_{E^-}|F\| \leq 1/2$ and $\|P_{E^+}|(L_{R-1} \cdot A_{R-1} \cdots L_0 \cdot A_0) \cdot F\| \leq 1/2$.

The proof will use the following lemma about dominated splittings.

Lemma 3.3. *There exists $C > 0$ such that if $\varepsilon > 0$ is sufficiently small, then the following holds. Let $L : \mathbb{R}^d \to \mathbb{R}^d$ be an invertible linear operator that preserves each of E^+, E^0 and E^-, and assume that for some $\lambda \in (0, 1/4)$ we have*

$$\|L|E^0\| \cdot \|L^{-1}|E^+\| \leq \lambda \quad \text{and} \quad \|L|E^-\| \cdot \|L^{-1}|E^0\| \leq \lambda.$$

Let $A \in \operatorname{supp} \mu_\varepsilon$ and let $F \subset \mathbb{R}^d$ be a d_0-dimensional subspace. Then:

1. *if $\|P_{E^-}|F\| \leq 1/2$ then $\|P_{E^-}|(L \cdot A) \cdot F\| \leq \lambda$;*
2. *if $\|P_{E^+}|(L \cdot A) \cdot F\| \leq 1/2$ then $\|P_{E^+}|F\| \leq \lambda$; and*
3. *if $\|P_{E^-}|F\|, \|P_{E^+}|(L \cdot A) \cdot F\| \leq \delta$, for some $\delta \in (0, 1/2)$, then*

$$\log \mathrm{Jac}_F(L \cdot A) < \log \mathrm{Jac}_{E^0}(L) + \log \mathrm{Jac}_{E^0}(P_{E^0} \cdot A) + C(\lambda + \delta).$$

Proof. If $v \in \mathbb{R}^d$ is a unit vector with $\|P_{E^-} \cdot v\|^2 \leq 1/2$, then

$$\|(P_{E^-} \cdot L) \cdot v\| = \|(L \cdot P_{E^-}) \cdot v\| \leq \lambda \, \|(L \cdot P_{E^+ \oplus E^0}) \cdot v\| = \lambda \, \|(P_{E^+ \oplus E^0} \cdot L) \cdot v\|.$$

Since $\varepsilon > 0$ is small, $\|P_{E^-}|F\| \leq 1/2$ implies $\|P_{E^-}|A \cdot F\|^2 \leq 1/2$. The first estimate follows.

Symmetrically if $v \in \mathbb{R}^d$ is a unit vector with $\|P_{E^+} \cdot v\|^2 \leq 1/2$, then

$$\|(P_{E^+} \cdot L^{-1}) \cdot v\| = \|(L^{-1} \cdot P_{E^+}) \cdot v\| \leq \lambda \, \|(L^{-1} \cdot P_{E^0 \oplus E^-}) \cdot v\| = \lambda \, \|(P_{E^0 \oplus E^-} \cdot L) \cdot v\|.$$

Since $\varepsilon > 0$ is small, $\|P_{E^+}|(L \cdot A) \cdot F\| \leq 1/2$ implies $\|P_{E^+}|L \cdot F\|^2 \leq 1/2$. The second estimate follows.

For any unit vector $v \in \mathbb{R}^d$ such that $\|P_{E^-} \cdot v\|^2 \leq 1/2$ and $\|P_{E^+} \cdot L \cdot v\| \leq \delta \|L \cdot v\|$,

$$\begin{aligned}\|L \cdot v - (P_{E^0} \cdot L) \cdot v\| &\leq \|(P_{E^+} \cdot L) \cdot v\| + \|(P_{E^-} \cdot L) \cdot v\| \\ &\leq \delta \|L \cdot v\| + \lambda \|(P_{E^+ \oplus E^0} \cdot L) \cdot v\| \leq (\delta + \lambda) \|L \cdot v\|.\end{aligned}$$

Thus if $F \subset \mathbb{R}^d$ satisfies $\|P_{E^-}|F\| \leq 1/2$ (and hence $\|P_{E^-}|A \cdot F\|^2 \leq 1/2$) and $\|P_{E^+}|(L \cdot A) \cdot F\| \leq \delta$, we can write $L|A \cdot F$ as $S_F \cdot L \cdot (P_{E^0}|A \cdot F)$, where $S_F : E^0 \to \mathbb{R}^d$ is a linear map with $\|S_F\| \leq (1 - \delta - \lambda)^{-1}$. We conclude that

$$\log \mathrm{Jac}_F(L \cdot A) \leq -d_0 \log(1 - \delta - \lambda) + \log \mathrm{Jac}_{E^0}(L) + \log \mathrm{Jac}_F(P_{E^0} \cdot A).$$

On the other hand, the function $\log \mathrm{Jac}_F(P_{E^0} \cdot A)$ is uniformly (on $A \in \operatorname{supp} \mu_\varepsilon$) Lipschitz as a function of those F satisfying $\|P_{E^+ \oplus E^-}|A \cdot F\| \leq 1/2$. Thus

$$|\log \mathrm{Jac}_F(P_{E^0} \cdot A) - \log \mathrm{Jac}_{E^0}(P_{E^0} \cdot A)| \leq C_0 \|P_{E^+ \oplus E^-}|F\|,$$

for some $C_0 > 0$. Since $\|P_{E^+ \oplus E^-}|F\| \leq \|P_{E^-}|F\| + \|P_{E^+}|F\| \leq \delta + \lambda$, the third estimate follows. □

Proof of Proposition 3.2. Define F_j, $0 \leq j \leq R$ by $F_0 = F$, $F_{j+1} = L_j \cdot A_j \cdot F$. First notice $\|P_{E^+}|F_R\| \leq 1/2$ and $\|P_{E^-}|F_0\| \leq 1/2$ imply, by iterated application of estimates (1-2) in the previous lemma, that $\|P_{E^+}|F_j\| \leq \lambda$ for $0 \leq j \leq R-1$, while $\|P_{E^-}|F_j\| \leq \lambda$ for $1 \leq j \leq R$. By item (3) in Lemma3.3 we get that $\log \mathrm{Jac}_{F_j}(L_j \cdot A_j) - (\log \mathrm{Jac}_{E^0}(L_j) + \log \mathrm{Jac}_{E^0}(P_{E^0} \cdot A_j))$ is at most $2C\lambda$ if $1 \leq j \leq R-2$, and at most $C\lambda + \frac{C}{2}$ for $j = 0$ or $j = R-1$. It follows that

$$\begin{aligned}&\log \mathrm{Jac}_F((L_{R-1} \cdot A_{R-1}) \cdots (L_0 \cdot A_0)) \leq \\ &\qquad \sum_{j=0}^{R-1} \log \mathrm{Jac}_{E^0}(L_j) + \sum_{j=0}^{R-1} \mathrm{Jac}_{E^0}(P_{E^0} \cdot A_j) + 2CR\lambda + C.\end{aligned}$$

If $0 < \lambda \leq (10C)^{-1} c(\varepsilon)$ and $R \geq 10Cc(\varepsilon)^{-1}$, this gives

$$\log \mathrm{Jac}_F((L_{R-1} \cdot A_{R-1}) \cdots (L_0 \cdot A_0)) \leq \sum_{j=0}^{R-1} \log \mathrm{Jac}_{E^0}(L_j) + \sum_{j=0}^{R-1} \mathrm{Jac}_{E^0}(P_{E^0} \cdot A_j) + \frac{3c(\varepsilon)}{10} R.$$

By the Law of Large Numbers, for every $\theta > 0$, if R is sufficiently large, the probability, with respect to $\mu_\varepsilon^{\otimes R}$, that

$$\frac{1}{R} \sum_{j=0}^{R-1} \mathrm{Jac}_{E^0}(P_{E^0} \cdot A_j) \geq -\frac{4c(\varepsilon)}{5}$$

is less than θ. The result follows. □

3.2. **Local.** In the second step, we explain how perturb along an orbit.

Proposition 3.4. *If $\varepsilon > 0$ is small, there exists $\lambda \in (0, 1/4)$ such that for every $\theta > 0$ there exists $R_0 \in \mathbb{N}$ with the following property. Let $R \geq R_0$, $N \geq R$, and let $f_j : (\mathbb{R}^d, 0) \to (\mathbb{R}^d, 0)$, $0 \leq j \leq N-1$, be germs of volume preserving diffeomorphisms such that the $L_j = Df_j(0)$ preserve E^+, E^0 and E^-, and such that*

$$\|L_j|E^0\| \cdot \|L_j^{-1}|E^+\| \leq \lambda \quad \textit{and} \quad \|L_j|E^-\| \cdot \|L_j^{-1}|E^0\| \leq \lambda.$$

Then for every small neighborhood U of $0 \in \mathbb{R}^d$, and $0 \leq j \leq N-1$, there exist measurable subsets Z_j of $U_j := f_{j-1} \circ \cdots \circ f_0(U)$, smooth volume preserving diffeomorphisms $\varphi_j : \mathbb{R}^d \to \mathbb{R}^d$ and perturbations $\tilde{f}_j := f_j \circ \varphi_j$ such that:

- $m(Z_j) \geq (1-2\theta)m(U_j)$,
- *φ_j coincides with* Id *outside U_j and $D\varphi_j(x) \in \operatorname{supp}\mu_\varepsilon$ for every $x \in \mathbb{R}^d$,*
- *for any $0 \leq j \leq N-R$, any $y \in Z_j$ and any d_0-dimensional space F satisfying $\|P_{E^-}|F\| \leq 1/3$ and $\|P_{E^+}|D(\tilde{f}_{j+R-1} \circ \cdots \circ \tilde{f}_j)(y) \cdot F\| \leq 1/3$, we have:*
$$\log \operatorname{Jac}_F(\tilde{f}_{j+R-1} \circ \cdots \circ \tilde{f}_j, y) \leq \operatorname{Jac}_{E^0}(L_{j+R-1} \circ \cdots \circ L_j) - \frac{c(\varepsilon)}{3}.$$

Thanks to the following lemma, it is possible to build a sequence of perturbations along an orbit that act like random perturbations.

Lemma 3.5. *Consider a sequence $f_j : U_j \to U_{j+1}$, $0 \leq j \leq N-1$, of C^1 volume preserving diffeomorphisms between bounded open sets of $\mathbb{R}^d$ and $f^j = f_{j-1} \circ \cdots \circ f_0$. Let ψ_j be volume preserving diffeomorphisms of $\mathbb{R}^d$ supported on the unit ball B. Let μ_j be the push-forward of normalized Lebesgue measure m on B under the map*
$$B \ni x \mapsto D\psi_j(x) \in \mathrm{SL}(d, \mathbb{R}).$$

Then there exist volume preserving diffeomorphisms φ_j of $\mathbb{R}^d$ such that, setting $\tilde{f}_j = f_j \circ \varphi_j$ and $\tilde{f}^j = \tilde{f}_{j-1} \circ \cdots \circ \tilde{f}_0$, we have:

1. *for $0 \leq j \leq N-1$, the diffeomorphism φ_j is arbitrarily C^0-close to the identity, equals* Id *outside U_j, and satisfies $D\varphi_j(x) \in \operatorname{supp}\mu_j$ for each $x \in \mathbb{R}^d$;*
2. *the push-forward of normalized Lebesgue measure m on U_0 under the map*
$$U_0 \ni x \mapsto (D\varphi_j(\tilde{f}^j(x)))_{j=0}^{N-1} \in \mathrm{SL}(d, \mathbb{R})^N$$
is arbitrarily close to $\mu_0 \otimes \cdots \otimes \mu_{N-1}$.

Proof. The proof is by induction on N. For $N = 0$ there is nothing to do. Assume it holds for $N-1$, and apply the result for the sequence $(f_j)_{j=0}^{N-2}$, yielding the sequence $(\varphi_j)_{0 \leq N-2}$. Define $\tilde{f}_j$ and $\tilde{f}^j$ as before, and let ν_{N-1} be the push-forward of normalized Lebesgue measure on U_0 under the map
$$H_{N-1} : U_0 \ni x \mapsto (D\varphi_j(\tilde{f}^j(x)))_{j=0}^{N-2} \in \mathrm{SL}(d, \mathbb{R})^{N-1},$$
so that ν_{N-1} is arbitrarily close to $\mu_0 \otimes \cdots \otimes \mu_{N-2}$.

For $n \in \mathbb{N}$, let $\{D_\ell^n\}_\ell$ be a finite family of disjoint closed balls in U_{N-1} such that:

- $\operatorname{diam}(D_\ell^n) < n^{-1}$;
- writing $D_\ell^n = \tilde{f}^{N-1}(\hat{D}_\ell^n)$, we have: $\sum_\ell m(\hat{D}_\ell^n) \geq (1 - n^{-1})\, m(U_0)$;
- if $x, y \in \hat{D}_\ell^n$ then $\|H_{N-1}(x) - H_{N-1}(y)\| \leq n^{-1}$.

Let $\xi_{n,\ell}$ be the conformal affine dilation that sends B into D_ℓ^n. Define $\varphi_{N-1,n}$ to be the identity outside $\bigcup_\ell D_\ell^n$ and by
$$\varphi_{N-1,n}(x) = \xi_{n,\ell}\psi_{N-1}(\xi_{n,\ell}^{-1}x), \quad x \in D_\ell^n.$$
Let $\nu_{N,n}$ be the push-forward of normalized Lebesgue measure on U_0 under
$$H_{N,n} : U_0 \ni x \mapsto (H_{N-1}, D\varphi_{N-1,n}(\tilde{f}^{N-1}(x))) \in \mathrm{SL}(d, \mathbb{R})^N.$$
The properties of the first item are immediate. Since ν_{N-1} is close to $\mu_0 \otimes \cdots \otimes \mu_{N-2}$, it is enough to show that $\lim_{n\to\infty} \nu_{N,n} = \nu_{N-1} \otimes \mu_{N-1}$ to establish the second

item. Equivalently, we must show that for a dense subset of compactly supported, continuous functions $\rho : \mathrm{SL}(d,\mathbb{R})^N \to \mathbb{R}$, we have

$$\lim_{n\to\infty} \int \rho \, d\nu_{N,n} = \int \rho \, d\nu_{N-1} \otimes d\mu_{N-1}. \tag{8}$$

Take ρ to be Lipschitz with constant C_ρ. Since $\mathrm{diam}(H_{N-1}(\hat{D}^n_\ell)) \leq n^{-1}$, the quantities

$$\frac{1}{m(\hat{D}^n_\ell)} \int_{\hat{D}^n_\ell} \rho(H_{N,n}(x)) dx,$$

and

$$\frac{1}{m(\hat{D}^n_\ell)^2} \int_{\hat{D}^n_\ell} \int_{\hat{D}^n_\ell} \rho(H_{N-1}(x), D\varphi_{N-1,n}(\tilde{f}^{N-1}(y))) \, dx \, dy$$

differ by at most $C_\rho n^{-1}$. By construction, for any y we have

$$\frac{1}{m(\hat{D}^n_\ell)} \int_{\hat{D}^n_\ell} \rho(H_{N-1}(x), D\varphi_{N-1}(\tilde{f}^{N-1}(y))) \, dy = \int \rho(H_{N-1}(x), z) \, d\mu_{N-1}(z),$$

so that

$$\begin{aligned} &\left| \int_{\bigcup_\ell \hat{D}^n_\ell} \rho(H_{N,n}(x)) \, dx - \int \int_{\bigcup_\ell \hat{D}^n_\ell} \rho(H_{N-1}(x), z) \, dx \, d\mu_{N-1}(z) \right| \\ &\qquad\qquad\qquad\qquad \leq C_\rho m\left(\bigcup_\ell \hat{D}^n_\ell \right) n^{-1}. \end{aligned} \tag{9}$$

Clearly

$$\left| \int \rho \, d\nu_{N,n} - \frac{1}{m(U_0)} \int_{\bigcup_\ell \hat{D}^n_\ell} \rho(H_{N,n}(x)) \, dx \right| \leq \|\rho\|_\infty n^{-1},$$

$$\left| \int \rho \, d\nu_{N-1} \otimes d\mu_{N-1} - \frac{1}{m(U_0)} \int \int_{\bigcup_l \hat{D}^n_\ell} \rho(H_{N-1}(x), z) \, dx \, d\mu_{N-1}(z) \right| \leq \|\rho\|_\infty n^{-1},$$

so that (9) implies (8). $\square$

Proof of Proposition 3.4. Use Proposition 3.2 to select λ, R_0 and W. Lemma 3.5 applied with $\psi_j = \psi^\varepsilon$ gives the φ_j. In particular, for every $0 \leq j \leq N - R$, there exists $Z_j \subset U_j$ with $m(Z_j) > (1 - 2\theta) m(U_j)$ such that the push forward by

$$U_0 \ni x \mapsto (D\varphi_n(\tilde{f}^n(x)))_{n=j}^{j+R-1} \in \mathrm{SL}(d,\mathbb{R})^N$$

of the set Z_j is arbitrarily close to W. It follows that if y is a point in Z_j and if F is a d_0-dimensional space satisfying $\|P_{E^-}|F\| \leq 1/3$ and $\|P_{E^+}|F'\| \leq 1/3$ for $F' = (L_{j+R-1} \cdot A_{j+R-1}) \cdots (L_j \cdot A_j) \cdot F$, then

$$\log \mathrm{Jac}_F((L_{j+R-1} \cdot A_{j+R-1}) \cdots (L_j \cdot A_j)) \leq \log \mathrm{Jac}_{E^0}(L_{j+R-1} \cdots L_j) - \frac{2c(\varepsilon)}{5} R,$$

where we denote $A_{j+i} = D\varphi_{j+i}(\tilde{f}_{j+i-1} \circ \cdots \circ \tilde{f}_j(y_j))$.

Since the f_i are diffeomorphisms, if the neighborhood U is small enough,

$$\log \mathrm{Jac}_F(D(\tilde{f}_{j+i-1} \circ \cdots \circ \tilde{f}_j)(y_j)) \leq \log \mathrm{Jac}_F((L_{j+R-1} \cdot A_{j+R-1}) \cdots (L_j \cdot A_j)) + \frac{c(\varepsilon)}{20} N.$$

The result follows. $\square$

3.3. **Global: proof of Theorem C'.** Using the local perturbation technique along orbits, we define in this third step the global perturbation by building towers with the following lemma.

Lemma 3.6. *Let $f : X \to X$ be an invertible bi-measurable map preserving a finite measure ν. Assume that periodic points have zero measure. Then for every $\ell_0 \in \mathbb{N}$, there exist measurable subsets $Y_\ell \subset X$, $0 \le \ell \le \ell_0 - 1$, such that the $f^k(Y_\ell)$ for $0 \le \ell \le \ell_0 - 1$, $0 \le k \le \ell_0 + \ell - 1$, are pairwise disjoint and their union has full measure.*

Proof. Note that it is enough to build an invariant measurable set $X' \subset X$ such that $\nu(X') \ge \nu(X/2)$ and such that the restriction of f and ν to X' satisfy the lemma. Considering a Rokhlin tower, there exists a measurable set Z disjoint from its $\ell_0 - 1$ first iterates, such that $X' = \cup_{\mathbb{Z}} f^n(Z)$ has measure larger than $\nu(X/2)$. Let $r : Z \to \mathbb{N}$ be the first return time to Z. Then set

$$Y_0 = \bigcup_{s\ge 0} f^{s\ell_0}\{x \in Z,\, r(x) \ge (s+2)\ell_0\}, \quad \text{and}$$

$$Y_j = \bigcup_{s\ge 0} f^{s\ell_0}\{x \in Z,\, r(x) = (s+1)\ell_0 + j\} \quad \text{for} \quad 1 \le j \le \ell_0 - 1.$$

□

Proof of Theorem C'. Let $B_\xi \subset \mathbb{R}^d$ be the ball centered at the origin of radius $\xi > 0$ small. We consider a precompact family of volume preserving smooth embeddings $\Psi_x : B_\xi \to M$, $x \in K$, such that $\Psi_x(0) = x$ and $D\Psi_x(0)$ sends E^+, E^0, E^- to $E_1(x)$, $E_2(x)$ and $E_3(x)$, respectively.

Let $\alpha > 0$ be small enough so that (from the dominated splitting $T_K M = E_1 \oplus E_2 \oplus E_3$) if F is α-close to $E_2(x)$ then for each $j \ge 0$ the image $Df^j(x).F$ is close to a subspace of $E_1(f^j(x)) \oplus E_2(f^j(x))$ and $Df^{-j}(x).F$ is close to a subspace of $E_2(f^{-j}(x)) \oplus E_3(f^{-j}(x))$. In particular for every $j \ge 0$,

$$\|P_{E^+}|\big(D\Psi_{f^{-j}(x)}(0)^{-1} \cdot Df^{-j}(x)\big) \cdot F\|,\ \|P_{E^-}|\big(D\Psi_{f^j(x)}(0)^{-1} \cdot Df^j(x)\big) \cdot F\| \ \le\ 1/5.$$

If $\mathcal{U}$ is small in the C^1-topology, for any $g \in \mathcal{U}$ and $j \ge 0$ we still have:

– if $g(x)$, $g^2(x)$, …, $g^j(x)$ are close enough to $f(x)$, $f^2(x)$, …, $f^j(x)$, then
$$\|P_{E^-}|\big(D\Psi_{f^j(x)}(0)^{-1} \cdot Dg^j(x)\big) \cdot F\| \ \le 1/4,$$
– if $g^{-1}(x)$,, …, $g^{-j}(x)$ are close enough to $f^{-1}(x)$, …, $f^{-j}(x)$, then
$$\|P_{E^+}|\big(D\Psi_{f^{-j}(x)}(0)^{-1} \cdot Dg^{-j}(x)\big) \cdot F\| \ \le 1/4.$$

We choose $\varepsilon > 0$ small (this choice depends on the neighborhood $\mathcal{U}$, see below) and apply Proposition 3.4 to get λ. The dominated splitting gives $J_0 \in \mathbb{N}$ such that for $x \in K$, the map $L_x = D\Psi_{f^{J_0}(x)}(0)^{-1} Df^{J_0}(x) D\Psi_x(0)$ satisfies

$$\|L_x|E^0\| \cdot \|L_x^{-1}|E^+\| \le \lambda \quad \text{and} \|L_x|E^-\| \cdot \|L_x^{-1}|E^0\| \le \lambda.$$

We then fix $\delta < c(\varepsilon)/(3J_0)$. Now take $\theta \in (0, \eta/10)$ and apply Proposition 3.4 to get R_0. Next, fix n_0 much larger than $J_0 \cdot R_0$. Given any $n \ge n_0$, set $R = [n/J_0] - 2$ and choose $\ell_0 \in \mathbb{N}$ much larger than n.

Since K has a dominated splitting, any periodic point $x \in K$ with period k satisfies $Df^k(p) \ne \mathrm{Id}$. The Implicit Function Theorem implies that the periodic points for f in K have measure 0. One can thus apply Lemma 3.6 to the restriction of f to K and obtain measurable sets $Y_\ell \subset K$, $0 \leq \ell \leq \ell_0 - 1$. Consider compact subsets $\hat{Y}_\ell \subset Y_\ell$ and small open neighborhoods $\hat{Y}_\ell \subset W_\ell \subset \operatorname{int} Q$ such that the sets $f^k(W_\ell)$, $0 \leq \ell \leq \ell_0 - 1$, $0 \leq k \leq \ell_0 + \ell - 1$, are pairwise disjoint and such that

$$m\bigg(K \setminus \bigcup_{\ell,k} f^k(\hat{Y}_\ell)\bigg) < \theta.$$

For $0 \leq \ell \leq \ell_0 - 1$, let $N_\ell = [\frac{\ell_0+\ell}{J_0}]$. For each $x \in \hat{Y}_\ell$, Proposition 3.4 applied to the sequence of diffeomorphisms $f_{j,x} := \Psi^{-1}_{f^{(j+1)J_0}(x)} \circ f^{J_0} \circ \Psi_{f^{jJ_0}(x)}$, $0 \leq j \leq N_\ell - 1$ and to the integers R, N gives a neighborhood D_x (which is the image $\Psi_x(U_x)$ of any small neighborhood U_x of 0), a sequence of diffeomorphisms $\varphi_{j,x}$, and a sequence of sets $Z_{j,x} \subset f^{jJ_0}(D_x)$ such that $m(Z_{j,x}) \geq (1-2\theta)m(D_x)$. By compactness, one can find finitely many such points $x_{\ell,s} \in \hat{Y}_\ell$ and reduce the associated neighborhoods $D_{\ell,s} := D_{x_{\ell,s}}$, such that the $D_{\ell,s}$ are pairwise disjoint subsets of W_ℓ and

$$m\bigg(K \setminus \bigcup_{\ell,k,s} f^k(\hat{D}_{\ell,s})\bigg) < 3\theta.$$

The domains $D_{\ell,s}$ may be chosen with small diameter so that for each point $z \in K$ in an iterate $f^{jJ_0}(D_{\ell,s})$, $0 \leq j \leq N_\ell$, and for any d_0-dimensional subspaces $F \subset T_zM$ and $F' \subset T_{f^{jR_0}(z)}M$,

$$\|P_{E^-}|D\Psi_z(0)^{-1} \cdot F\| \leq 1/4 \ \Rightarrow\ \|P_{E^-}|D\Psi_{f^{jJ_0}(x_{\ell,s})}(0)^{-1} \cdot F\| \leq 1/3 \tag{10}$$

$$\|P_{E^+}|D\Psi_{f^{RJ_0}(z)}(0)^{-1} \cdot F'\| \leq 1/4 \ \Rightarrow\ \|P_{E^+}|D\Psi_{f^{(j+R)J_0}(x_{\ell,s})}(0)^{-1} \cdot F'\| \leq 1/3$$

Let us define the diffeomorphism φ in each $f^{jJ_0}(D_{\ell,s})$, $0 \leq j \leq N_\ell - 1$ by

$$\varphi = \Psi_{f^{jJ_0}(x_{\ell,s})} \circ \varphi_{j,x_{\ell,s}} \circ \Psi^{-1}_{f^{jJ_0}(x_{\ell,s})},$$

and let $\varphi = \mathrm{Id}$ otherwise. It is clear that if the neighborhoods U_x are chosen small enough, then φ is arbitrarily close to the identity for the topology C^0. Also, if ε is small enough then φ is close to the identity for the topology C^1. We set $g = f \circ \varphi$.

Define the set Λ to be the set of all points y belonging to some $f^k(D_{\ell,s})$, with $0 \leq k \leq (N_\ell - 1)J_0 - n$, such that $f^{jJ_0-k}(y) \in Z_{j,x_{\ell,s}}$, where $j = [k/J_0] + 1$. Hence

$$k \leq jJ_0 \leq (j+R)J_0 \leq k + n.$$

Clearly, if ℓ_0 is large and since $10\theta < \eta$, we have $m(K \setminus \Lambda) < \eta$.

Now consider $y \in \Lambda \cap K \cap g^{-n}(K)$ and a d_0-dimensional subspace $F \subset T_yM$ that is α-close to $E_2(f,y)$ and whose image $Dg^n.F$ is α-close to $E_2(f, g^n(y))$. We also introduce $j, k, x_{\ell,s}$ as defined above such that $f^{jJ_0-k}(y)$ belongs to $Z_{j,x_{\ell,s}}$. Since $k - jJ_0$ and $(j+R)J_0 - (k+n)$ are bounded (by $2J_0$) and g can be chosen arbitrarily close to f for the C^1-topology, by the choice of α we have

$$\|P_{E^-}|D\Psi_{f^{jJ_0-k}(y)}(0)^{-1} \cdot D^{jJ_0-k}g(y) \cdot F\| \leq 1/4,$$

$$\|P_{E^+}|D\Psi_{f^{(j+R)J_0-k}(g^n(y))}(0)^{-1} \cdot Dg^{(j+R)J_0-k}(y) \cdot F\| \leq 1/4.$$

By (10), this gives:

$$\|P_{E^-}|D\Psi_{f^{jJ_0}(x_{\ell,s})}(0)^{-1} \cdot D^{jJ_0-k}g(y) \cdot F\| \leq 1/3,$$

$$\|P_{E^+}|D\Psi_{f^{(j+R)J_0}(x_{\ell,s})}(0)^{-1} \cdot Dg^{(j+R)J_0-k}(y) \cdot F\| \leq 1/3.$$

Let $F' = D^{jJ_0-k}g(y).F$. Since $f^{jJ_0-k}(y)$ belongs to $Z_{j,x_{\ell,s}}$, by applying Proposition 3.4 we obtain:

$$\log \mathrm{Jac}_{F'}(g^{RJ_0}, g^{jJ_0-k}(y)) \leq \log \mathrm{Jac}_{E_2(f,f^{jJ_0}(x_{\ell,s}))}(f^{RJ_0}, f^{jJ_0}(x_{\ell,s})) - \frac{c(\varepsilon)}{3}R + 4C_0,$$

where C_0 bounds $|\log \mathrm{Jac}_H(D\Psi_x)|$ for any $x \in K$ and any d_0-dimensional space H.

If g is sufficiently C^0-close to f, and if the sets $D_{\ell,s}$ have small diameter, then the orbits $(f^{-k}(y), \dots, f^{2\ell_0-k}(y))$ and $(x_{\ell,s}, \dots, f^{2\ell_0}(x_{\ell,s}))$ are arbitrarily close. It follows that there exists a constant $C_1 > 0$, which depends on J_0 but not on R, such that:

$$\log \mathrm{Jac}_F(g^n, y) \leq \log \mathrm{Jac}_{E_2(f,y)}(f^n, y) - \frac{c(\varepsilon)}{3J_0}n + 4C_0 + C_1.$$

If n_0 (hence n) has been chosen large enough, one gets (7) by our choice of δ. This ends the proof of Theorem C'. $\square$

4. Stable Ergodicity

We now build blenders (Theorem F) assuming Theorems A, D, and E. We then obtain Theorem B using the following criterion: (similar to [RRTU]):

Partial hyperbolicity + accessibility + stable/unstable blenders

+ positive measure sets of points with large stable (resp. unstable) dimension

$\Rightarrow$ *Ergodicity.*

4.1. Regularization of C^1-diffeomorphisms.

The proof of Theorem B uses Theorem A, and hence forces us to work with diffeomorphisms that are only C^1. To recover results for C^r-diffeomorphisms, $r > 1$, we will use:

Theorem 4.1 (Avila [Av])**.** *$\mathrm{Diff}^\infty_m(M)$ is dense in $\mathrm{Diff}^1_m(M)$.*

4.2. Criterion for ergodicity.

If O is a hyperbolic periodic orbit, we define the following sets (with the notation introduced in Section 2.1.4):

$$H^s_{\mathrm{Pes}}(O) = \{x \text{ Oseledets regular } : W^-(x) \pitchfork W^u(O) \neq \emptyset\},$$
$$H^u_{\mathrm{Pes}}(O) = \{x \text{ Oseledets regular } : W^+(x) \pitchfork W^s(O) \neq \emptyset\},$$

where $W_1 \pitchfork W_2$ denotes the set of transverse intersection between manifolds W_1, W_2, i.e. the set of points x such that $T_xW_1 + T_xW_2 = T_xM$. The *Pesin homoclinic class* is $H_{\mathrm{Pes}}(O) := H^s_{\mathrm{Pes}}(O) \cap H^u_{\mathrm{Pes}}(O)$. We stress the fact that $H^s_{\mathrm{Pes}}(O)$ can contain points x whose stable dimension $\dim(E^-(x))$ is strictly larger than the stable dimension of O. However the set $H_{\mathrm{Pes}}(O)$ only contains non-uniformly hyperbolic points whose stable/unstable dimensions are the same as O.

As a consequence of [Ka] (see also [KH, section 20]), we have:

Theorem 4.2 (Katok)**.** *Let $r > 1$ and $f \in \mathrm{Diff}^r(M)$. Let μ be a hyperbolic invariant probability (μ-almost every point has no zero Lyapunov exponent). Then there exist (at most) countably many Pesin homoclinic classes $H_{Pes}(O_n)$ whose union has full μ-measure.*

In the previous statement the restriction $\mu|H_{\mathrm{Pes}}(O_n)$ is not ergodic in general. In the case μ is smooth this is however always the case.

Theorem 4.3 (Rodriguez-Hertz - Rodriguez-Hertz - Tahzibi - Ures [RRTU])**.** *Let $f \in \mathrm{Diff}^r_m(M)$ with $r > 1$ and let O be a hyperbolic periodic point such that $m(H^s_{Pes}(O))$ and $m(H^u_{Pes}(O))$ are positive. Then $H^s_{Pes}(O), H^u_{Pes}(O), H_{Pes}(O)$ coincide m-almost everywhere and $m|H_{Pes}(O)$ is ergodic.*

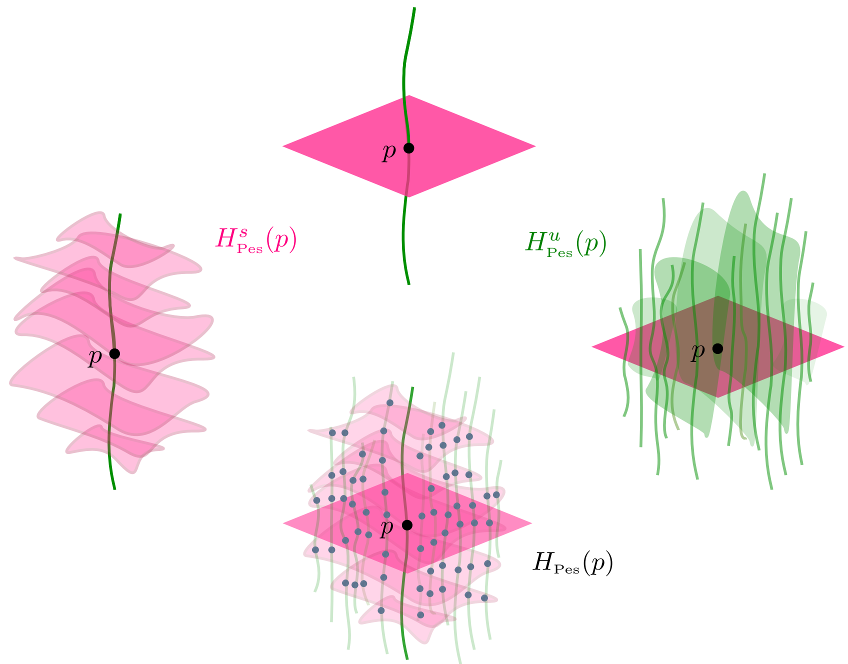

FIGURE 4. The Pesin homoclinic class

Having introduced the notion of the Pesin homoclinic class, we can now prove Corollary A".

Proof of Corollary A". Recall that $f \in \mathrm{Diff}^1_m(M)$ is weakly mixing if and only if $f \times f$ is ergodic with respect to $m \times m$.

Given a continuous function $\phi : M \times M \to \mathbb{R}$ and $\epsilon > 0$, we denote by $\mathcal{U}(\phi, \epsilon)$ the set of all $f \in \mathrm{Diff}^1_m(M)$ such that, for some $n \geq 1$, the set of all $(x, y) \in M \times M$ satisfying

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k(x), f^k(y)) - \int \phi(x, y) dm(x) dm(y) \right| < \epsilon$$

has measure strictly larger than $1 - \epsilon$. Note that $\mathcal{U}(\phi, \epsilon)$ is open, and that for any dense subset $\Omega \subset C^0(M \times M, \mathbb{R})$, $f \times f$ is ergodic if and only if f belongs to $\bigcap_{\phi \in \Omega} \bigcap_{\epsilon > 0} \mathcal{U}(\phi, \epsilon)$.

Let us say that f is *ϵ-weak mixing* if it admits an invariant subset X of measure strictly larger than $1 - \epsilon$, such that $f|X$ is weak mixing. Notice that if f is ϵ-weak mixing then $f \in \mathcal{U}(\phi, 2\epsilon - \epsilon^2)$ for every $\phi \in C^0(M \times M, \mathbb{R})$. Thus to prove the genericity statement of Corollary A", it is enough to prove that ϵ-weak mixing is dense among the diffeomorphims in $\mathrm{Diff}^1_m(M)$ with positive metric entropy.

Let $f \in \mathrm{Diff}^1_m(M)$ be a C^1-generic diffeomorphism given by Theorem A. We may assume that f has the following additional C^1-generic properties:

(1) f is topologically transitive, by [BC, Théorème 1.3],
(2) for any hyperbolic periodic point p, we have $W^u(p) \cap W^s(f(p)) \neq \emptyset$ and the intersection is transverse, by [AC, Theorems 3 and 4], and
(3) there exist hyperbolic periodic points $p_1, \dots, p_k$ such that for every $\epsilon > 0$ and every $g \in \mathrm{Diff}^2_m(M)$ sufficiently C^1-close to f, there exists a p_i such that the continuation $O(p_i(g))$ of the orbit of p_i has the following property: the Pesin homoclinic class $H^s_{\mathrm{Pes}}(O(p_i(g)))$ has m-measure $> 1 - \epsilon$ and is ergodic and nonuniformly Anosov, by [AB, Lemma 5.1].

Let $p_1, \dots, p_k$ be given by item (3) and let $\epsilon > 0$. If $g \in \mathrm{Diff}^2_n(M)$ is sufficiently C^1-close to f, then: by item (2) for each $i = 1, \dots, k$, there still exists a transverse intersection point between $W^u(p_i(g))$ and $W^s(g(p_i(g)))$ associated to the hyperbolic continuation $p_i(g)$. By item (3) there exists an $i \in 1, \dots, k$ such that the Pesin homoclinic class $H^s_{\mathrm{Pes}}(O(p_i(g)))$ has m-measure $> 1 - \epsilon$ and is ergodic and nonuniformly Anosov.

It follows from [P] that $H^s_{\mathrm{Pes}}(O(p_i(g)))$ decomposes as a disjoint union of measurable sets $A \cup g(A) \cup \dots \cup g^{\ell-1}(A)$ and that the restriction $m|A$ is Bernoulli for g^ℓ. On the other hand, since $W^u(p_i(g)) \cap W^s(g(p_i(g))) \neq \emptyset$, the Pesin homoclinic class of the orbits of $p_i(g)$ for g and g^ℓ coincide, implying by Theorem 4.3 that $m|H^s_{\mathrm{Pes}}(O(p_i(g)))$ is ergodic for g^ℓ. This shows that $\ell = 1$, and that g is Bernoulli, and in particular weakly mixing, on $H^s_{\mathrm{Pes}}(O(p_i(g)))$. Thus g is ϵ-weakly mixing, and so ϵ-weak mixing is dense in $\mathrm{Diff}^1_m(M)$. This completes the proof. $\square$

From Theorem 4.3 we obtain a criterion for the global ergodicity of the volume, which we will use to prove Theorem C and its sequelae.

Corollary 4.4. *Let $f \in \mathrm{Diff}^r_m(M)$ with $r > 1$ such that:*

- *f preserves a partially hyperbolic splitting $TM = E^{uu} \oplus E^c \oplus E^{ss}$ and a dominated splitting $TM = E_1 \oplus E_2$ such that $E^{uu} \subset E_1 \subset (E^{uu} \oplus E^c)$.*
- *There exists a horseshoe Λ with unstable bundle $E_1|\Lambda$ and which is both a $(d_{uu} + d_c)$-unstable and $(d_c + d_{ss})$-stable blender, where $d_* = \dim E^*$,*
- *The orbit of m-almost every point is dense in M.*
- *There exist a positive m-measure set of regular points x having unstable dimension $\dim(E^+(x)) \geq \dim(E_1)$ and a positive m-measure set of regular points having stable dimension $\dim(E^-(x)) \geq \dim(E_2)$.*

Then f is ergodic.

Proof. Let us consider two charts φ^u, φ^s centered at two points $x^u, x^s \in \Lambda$ as in the definition of unstable and stable blenders given at Section 1.4. Let O be a periodic orbit in Λ.

By assumption the orbit of m-almost every point $x \in M$ is dense in M, and accumulates on x^u. By continuity of the leaves of the strong stable foliation in the C^1 topology, the strong stable manifold $W^{ss}_{loc}(f^n(x))$ for some $n \in \mathbb{Z}$ is arbitrarily C^1-close to $W^{ss}_{loc}(x^u)$. From the blender property, we deduce that $W^{ss}_{loc}(f^n(x))$ intersects $W^u(y)$ for some point $y \in \Lambda$. Since M has a global dominated splitting $E_1 \oplus E_2$, if the stable dimension $\dim(E^-(x))$ of x is greater than or equal to the stable dimension $\dim(E_2)$ of Λ, the stable manifold of x intersects $W^u(y)$ transversely. Since the unstable manifold of O is dense in the unstable set of Λ, this implies that x belongs to $H^s_{\mathrm{Pes}}(O)$.

Similarly, m-almost every point whose unstable dimension is greater than or equal to E_1 belongs to $H^u_{\mathrm{Pes}}(O)$. Note that m-almost every point has either stable dimension $\geq \dim(E_2)$ or unstable dimension $\geq \dim(E_1)$. Consequently the union $H^u_{\mathrm{Pes}}(O) \cup H^s_{\mathrm{Pes}}(O)$ has full volume. By our last assumption, $H^s_{\mathrm{Pes}}(O)$ and $H^u_{\mathrm{Pes}}(O)$ both have positive m-measure. Theorem 4.3 thus applies and $H^s_{\mathrm{Pes}}(O), H^u_{\mathrm{Pes}}(O), M$ coincide up to a set of zero-volume. Moreover $m = m|H_{\mathrm{Pes}}(O)$ is ergodic. $\square$

4.3. **Proof of Theorem F.** Consider a diffeomorphism $f \in \mathrm{Diff}^1_m(M)$ that preserves a non-trivial dominated splitting $TM = E \oplus F$. For diffeomorphisms C^1-close to f this splitting persists, and in particular the first case of Theorem A does not hold. It follows that there exists $f_1 \in \mathrm{Diff}^1_m(M)$ close to f that is ergodic and non-uniformly Anosov. We can thus change the dominated splitting so that $\dim(E)$ coincides with the stable dimension of m-almost every point.

We can furthermore require that f_1 belongs to the dense G_δ sets provided by Theorem 2.3 and Corollary 2.4. In particular, for any diffeomorphism f_2 in a C^1-neighborhood $\mathcal{U} \subset \mathrm{Diff}^1_m(M)$ of f_1, the set of non-uniformly hyperbolic points whose unstable dimensions coincide with $\dim(E)$ has positive volume. By Theorems 4.1, 4.2 and 4.3, one thus can choose

- a C^2 diffeomorphism f_2 that is C^1-close to f_1,
- a hyperbolic periodic orbit O for f_2, such that $m(H_{\mathrm{Pes}}(O)) > 0$ and the unstable dimension of O is $\dim(E)$.

Pesin's formula [P] now applies to the normalization μ of $m|H_{\mathrm{Pes}}(O)$:

Theorem 4.5 (Pesin)**.** *If $f \in \mathrm{Diff}^1(M)$ and μ is an ergodic invariant probability measure absolutely continuous with respect to a volume of M, then the Lyapunov exponents $\lambda^1_\mu \geq \cdots \geq \lambda^d_\mu$ of μ counted with multiplicity satisfy:*

$$h_\mu(f) = \sum_i \max(\lambda^i_\mu, 0).$$

For any $\delta > 0$, Theorem D provides us with a C^2-diffeomorphism f_3 that is C^1-close to f_2 and with an affine horseshoe Λ whose linear part is constant and equal to a diagonal matrix $A = \mathrm{Diag}(\exp(\lambda^1), \dots, \exp(\lambda^d))$, such that

$$h_{top}(\Lambda, f_3) \geq \sum_i \max(\lambda^i, 0) - \delta.$$

By Theorem E, one deduces that there exists $f_4 \in \mathrm{Diff}^1_m(M)$ that is C^1-close to f_3 such that the hyperbolic continuation of Λ is a $(d-1)$-unstable blender. Applying again Theorems D and E to the measure of maximal entropy of Λ for f_4, one builds a diffeomorphism g that is C^1-close to f_4 (hence to the initial diffeomorphism f) such that the continuation of Λ is a $(d-1)$-dimensional stable blender Λ, proving Theorem F.

4.4. **Metric transitivity.** Using that accessibility of the strong distributions for partially hyperbolic diffeomorphisms is C^1-open and dense [DW], Brin's argument [Br] gives:

Theorem 4.6 (Brin, Dolgopyat-Wilkinson)**.** *For any partially hyperbolic diffeomorphisms in an open and dense subset of* $\mathrm{Diff}^1_m(M)$*, m-almost every point has a dense orbit in* M*.*

4.5. **Proof of Theorem B.** For $r > 1$, consider the C^1-open set $\mathcal{PH}^r_m(M)$ of diffeomorphisms $f \in \mathrm{Diff}^r_m(M)$ that preserve a partially hyperbolic decomposition $TM = E^s \oplus E^c \oplus E^u$. By Theorems 4.1, 4.6 and Theorem F, there exists a C^1-dense and C^1-open subset $\mathcal{U} \subset \mathcal{PH}^r_m(M)$ of diffeomorphisms f having a horseshoe Λ that is both a $(d_{uu} + d_c)$-dimensional unstable blender and a $(d_c + d_{ss})$-dimensional stable blender and such that m-almost every orbit is dense. Moreover there exists a dominated splitting $TM = E \oplus F$ such that $\dim(E)$ coincides with the stable dimension of Λ and the set of non-uniformly hyperbolic points whose unstable dimension equals $\dim(E)$ has positive volume. By Corollary 4.4, any diffeomorphism in the open set $\mathcal{U}$ is ergodic, proving Theorem B. Since the set Nuh_f has positive volume, the measure m is hyperbolic and the diffeomorphism f is non-uniformly Anosov. By [P, Theorem 8.1], the system (f, m) is Bernoulli.

5. Horseshoes with simple dominated spectrum

Our goal in this section is to prove Theorem 1.4, which allows us to extract from a horseshoe Λ a subhorseshoe $\widetilde{\Lambda}$ that has a dominated splitting into one-dimensional subbundles, after an arbitrarily C^1-small perturbation.

Here is the scheme of the proof. After a C^1-small perturbation, we can assume that the given diffeomorphism is smooth in a neighborhood of Λ. The initial step the proof of this result is to apply Katok's Theorem 1.5 to the measure of maximal entropy of Λ. This immediately implies Theorem 1.4 when the Lyapunov spectrum of μ is simple, so our basic task will be to eliminate multiplicities in the Lyapunov spectrum.

5.1. **Non-triviality of the Lyapunov spectrum for cocycles over subshifts.** In this section we recall some basic results of [BGV].

Let $\sigma : \Sigma \to \Sigma$ be a *subshift*, i.e., the restriction of the shift on $\mathcal{A}^{\mathbb{Z}}$ (where $\mathcal{A}$ a finite set) to a transitive invariant compact subset. For $l \le r$ integers, we define the (l, r)*-cylinder* containing $x \in \Sigma$ as the set of all $y \in \Sigma$ such that $\pi_j(y) = \pi_j(x)$ for $l \le j \le r$, where $\pi_j : \Sigma \to \mathcal{A}$ are the coordinate projections. We say that x and y have the same *stable set* (resp. *local stable set*) if $\pi_i(x) = \pi_i(y)$ for any large integer i (resp. for any $i \ge 0$).

A subshift is called *Markovian* if there exists a directed graph $\mathcal{G}$ with vertices in $\mathcal{A}$ such that Σ consists of all sequences corresponding to directed bi-infinite paths in $\mathcal{G}$. A *subshift of finite type* is a subshift which is topologically conjugate to a Markovian subshift.

Let $\sigma : \Sigma \to \Sigma$ be a Markovian subshift and let $A : \Sigma \to \mathrm{GL}(d, \mathbb{R})$ be continuous. We say that the cocycle (σ, A) has *stable holonomies* if for every x, y in the same stable set there exists $H_s(x, y) \in \mathrm{GL}(d, \mathbb{R})$ such that

(1) $H_s(y, z) \circ H_s(x, y) = H_s(x, z)$,
(2) $H_s(\sigma(x), \sigma(y)) \circ A(x) = A(y) \circ H_s(x, y)$,
(3) $(x, y) \mapsto H_s(x, y)$ is a continuous function restricted to the set of $(x, y) \in \Sigma \times \Sigma$ such that y belongs to the local stable manifold of x.

We define analogously the unstable holonomies $H_u(x, y)$.

We now give a condition for deducing the existence of stable holonomies.

Proposition 5.1 (Lemme 1.12 in [BGV]). *If there exists $C, \epsilon > 0$ such that whenever $x, y \in \Sigma$ belong to the same local stable manifold we have for every $n \geq 0$,*

$$\|A_n(x)\|\|A_n(x)^{-1}\|\|A(\sigma^n(x)) - A(\sigma^n(y))\| < Ce^{-\epsilon n}, \tag{11}$$

then the cocycle admits stable holonomies which satisfy:

$$H_s(x,y) = \lim_{n\to+\infty} A^{-1}(y)\dots A(\sigma^n(y))^{-1}A(\sigma^n(x))\dots A(x). \tag{12}$$

The following is a particular case (for the measure of maximal entropy) of the criterion for non-degenerate Lyapunov spectrum in [BGV].

Theorem 5.2 (Bonatti - Goméz-Mont - Viana). *Assume that the cocycle (σ, A) has stable and unstable holonomies and that its Lyapunov exponents with respect to the measure of maximal entropy of (Σ, σ) are all the same. Then there exists a continuous family μ_x, $x \in \Sigma$, of probability measures on $P\mathbb{R}^d$ such that*

$$A(x)_*(\mu_{\sigma(x)}) = \mu_x,\ \ H_s(x,y)_*(\mu_y) = \mu_x \text{ and } H_u(x,y)_*(\mu_y) = \mu_x.$$

The following lemma will allow us to show that the conclusion of Theorem 5.2 is not satisfied (hence that the Lyapunov exponents do not coincide).

Lemma 5.3. *For $d \geq 2$ and (B, B') in a dense G_δ subset of $PGL(d,\mathbb{R})\times PGL(d,\mathbb{R})$, there is no probability measure on $P\mathbb{R}^d$ which is invariant by both B and B'.*

Proof. For (B, B') in a dense G_δ subset of $PGL(d,\mathbb{R}) \times PGL(d,\mathbb{R})$,

- the Oseledets spaces one or two-dimensional,
- the argument of complex eigenvalues is not a rational multiple of 2π,
- if the Oseledets splitting of B or B' is not trivial, then B and B' have distinct Oseledets subspaces,
- if $d = 2$ and B, B' have complex eigenvalues, they do not belong to the same compact subgroup of $PGL(2,\mathbb{R})$.

The two first items imply that the ergodic B-invariant measure on $P\mathbb{R}^d$ are Dirac measures along the one-dimensional Oseledets subspaces and smooth measures along the 2-dimensional Oseledets spaces. The same holds for B'. By the third item, there is no probability measure simultaneously invariant by B and B' if the Oseledets splitting of B or B' is not trivial.

If the Oseledets splitting of B and B' is trivial, then $d = 2$ and B, B' have complex eigenvalues. The set of elements of $PGL(2,\mathbb{R})$ that preserve the (unique) probability measure on $P\mathbb{R}^2$ that is B-invariant is precisely the compact subgroup of $PGL(2,\mathbb{R})$ containing B. Consequently B and B' do not preserve the same measure on $P\mathbb{R}^2$. □

5.2. **Eliminating multiplicities in the Lyapunov spectrum.** Let $f : M \to M$ be a C^1 diffeomorphism, and let Λ be a horseshoe with a dominated splitting $TM = E_1 \oplus \dots \oplus E_\ell$. Recall [An2] that it is topologically conjugate to a subshift of finite type Σ by a homeomorphism $h\colon \Lambda \to \Sigma$. Using local smooth charts, the restriction of the derivative cocycle Df to any subbundle E_j can be represented by a continuous $GL(d,\mathbb{R})$-cocycle A_j on Σ.

We say that the bundle E_j is *α-pinched* if there is $n \geq 1$ such that for any $x \in K$

$$\|Df^n(x)|E_j(x)\|\|(Df^n(x)|E_{j+1}(x))^{-1}\|\|Df^n(x)\|^\alpha < 1,$$
$$\|(Df^n(x)|E_j(x))^{-1}\|\|Df^n(x)|E_{j-1}(x)\|\|(Df^n(x))^{-1}\|^\alpha < 1.$$

It has the following well-known consequence (see for instance [PSW]).

Proposition 5.4. *If E_j is α-pinched and f is C^2, then E_j is α-Hölder.*

We say that the bundle E_j is *α-bunched* if there is $n \geq 1$ such that for any $x \in K$

$$\|Df^n(x)|E_j(x)\|\|(Df^n(x)|E_j(x))^{-1}\|\|Df^n(x)|E^s(x)\|^\alpha < 1,$$

$$\|Df^{-n}(x)|E^u(x)\|^\alpha\|Df^{-n}(x)|E_j(x)\|\|(Df^{-n}(x)|E_j(x))^{-1}\| < 1.$$

Note that if f is C^2 and if E_j is α-pinched and α-bunched, then the condition (11) is satisfied and by Proposition 5.1, A_j has stable and unstable holonomies.

Theorem 5.5. *Let f be a C^k diffeomorphism and Λ a horseshoe with a dominated splitting $TM = E_1 \oplus \cdots \oplus E_\ell$. If E_j is α-pinched and α-bunched with $\dim E_j \geq 2$, then in every C^k-neighborhood of f, there exists g with the following property. The Lyapunov exponents of Dg along $E_j(g)$ with respect to the measure of maximal entropy of the continuation Λ_g are not all equal.*

Moreover if f is volume preserving, g can be chosen volume preserving as well.

Proof. The α-pinching and α-bunching are robust. Up to a C^k-perturbation, one can thus assume that f is smooth, hence the cocycle A_j associated to $Df|E_j$ admits stable and unstable holonomies H_s, H_u. Let $p \in \Sigma$ be a n-periodic point and $q \in \Sigma$ a homoclinic point of p (so that $\sigma^{\pm\ell n}(q) \to p$ as $\ell \to \infty$). We set $d = \dim(E_j)$.

The G_δ set $\mathcal{G} \subset PGL(d, \mathbb{R}) \times PGL(d, \mathbb{R})$ of Lemma 5.3 can be obtained as a union

$$\mathcal{G} = \bigcup_{B \in \mathcal{G}_1} \{B\} \times \mathcal{G}_B,$$

where $\mathcal{G}_1$ and each $\mathcal{G}_B$ is a dense G_δ subset of $PGL(d, \mathbb{R})$. Perturbing f near $h^{-1}(p)$, if necessary, we may assume that $B = A(\sigma^{n-1}(p)) \circ \cdots \circ A(p)$ belongs to $\mathcal{G}_1$. Consider another perturbation of f near $h^{-1}(q)$ and away from the closure of $\{f^\ell(h^{-1}(q))\}_{\ell \in \mathbb{Z} \setminus \{0\}}$, such that $g \circ h^{-1}(q) = f \circ h^{-1}(q)$. Let us consider $N \geq 1$ large such that $\sigma^N(q)$ and $\sigma^{-N}(q)$ belong to the local stable manifold and to the local unstable manifold of p respectively. The formula (12) shows that the holonomies $H_u(p, \sigma^{-N}(q))$ and $H_s(\sigma^N(q), p)$ are not modified by the perturbation near q. One can thus assume that after the perturbation the following map belong to $\mathcal{G}_B$:

$$B' = H_s(\sigma^N(q), p) \circ \big(A(\sigma^{N-1}(q)) \circ \dots A(q) \circ \dots A(\sigma^{-N}(q))\big) \circ H_u(p, \sigma^{-N}(q)).$$

Since there is no probability measure on $P\mathbb{R}^d$ that is simultaneously preserved by B and B', Theorem 5.2 shows that the Lyapunov exponents along E_j for the measure of maximal entropy on Λ can not all coincide. $\square$

5.3. **Proof of Theorem 1.4.** Up to an arbitrarily small perturbation, we can assume that f is smooth in a neighborhood of Λ. Let ℓ be the number of distinct Lyapunov exponents of μ. Using Theorem 1.5, we can replace Λ by a sub horseshoe Λ_1 endowed with a dominated splitting into ℓ subbundles and whose topological entropy is arbitrarily close to the entropy of Λ. If $\ell = \dim(M)$, we are done.

If $\ell < \dim(M)$, we apply Theorem 5.5 to obtain a perturbation f_1 of f for which the measure of maximal entropy on the continuation Λ_1' of Λ_1 has $\ell_1 > \ell$ distinct Lyapunov exponents. Theorem 1.4 follows after repeating this procedure at most $d - \ell$ times. $\square$

6. A linear horseshoe by perturbation

In this section we prove Theorem 1.3.

6.1. **Partially hyperbolic horseshoes with essential center bundle.** If Λ is a horseshoe, it admits a unique measure of maximal entropy μ. We refer to [Bow] for its properties. In particular:

- The measure μ may be disintegrated along *every* unstable manifold $W^u(x)$, $x \in \Lambda$ as a (non-finite) measure μ^u, which is well-defined up to a multiplicative constant. Hence the notion of measurable sets $A, B \subset W^u(x)$ with positive μ^u-measure is well defined, as is their ratio $\mu^u(A)/\mu^u(B)$.
- With respect to the disintegration μ^u, the map f has constant Jacobian along unstable leaves: for any measurable sets $A, B \subset W^u(f(x))$, the ratios $\mu^u(A)/\mu^u(B)$ and $\mu^u(f^{-1}(A))/\mu^u(f^{-1}(B))$ are equal.
- The disintegration μ^u is invariant under stable holonomy. If ρ is small enough, for any $x, y \in \Lambda$ with $y \in W^s(x, \rho)$ the stable holonomy defines a map $\Pi^s_{x,y}$ from $W^u(x, \rho)$ to $W^u(y)$: the point $\Pi^s_{x,y}(z)$ is the unique intersection point between $W^s_{loc}(z)$ and $W^u_{loc}(y)$. Then for any two measurable sets $A, B \subset W^u(x, \rho)$, the ratios $\mu^u(A)/\mu^u(B)$ and $\mu^u(\Pi^s_{x,y}(A))/\mu^u(\Pi^s_{x,y}(B))$ are equal.

In general, the strong unstable leaves have zero μ^u-measure, as the next proposition makes precise.

Proposition 6.1. *Let Λ be a horseshoe for a C^1-diffeomorphism f with a partially hyperbolic splitting:*
$$T_\Lambda M = E^{uu} \oplus E^c \oplus E^s,$$
where E^s is the stable bundle and $E^u = E^{uu} \oplus E^c$ is the unstable bundle in the hyperbolic splitting for $f|_\Lambda$. Then the following dichotomy holds.

(1) *Either $\mu^u(W^{uu}(x)) = 0$, for every $x \in \Lambda$, where μ^u is the disintegration of the measure of maximal entropy along $W^u(x)$,*
(2) *or $\Lambda \cap W^u(x) \subset W^{uu}(x)$ for every $x \in \Lambda$.*

In the second case, note that the local stable and strong unstable laminations are jointly integrable and that the Hausdorff dimension of $\Lambda \cap W^u(x)$ is therefore less than or equal to $\dim(E^{uu})$.

When the first case holds, we say that the center bundle E^c of Λ is *essential*.

Proof of Proposition 6.1. Consider a Markov partition $\mathcal{C} = \{C_0, \dots, C_s\}$ of Λ into small compact disjoint rectangles: in particular, there exists ρ such that the local manifolds $W^u(x, \rho)$ and $W^s(y, \rho)$ intersect at a unique point whenever x, y belong to the same rectangle C_i. We denote by $C(x)$ the rectangle containing the point $x \in \Lambda$. For $n \geq 1$, we also introduce the iterated Markov partition $\mathcal{C}^n$, which is the collection of rectangles of the form $C_{i_0} \cap f^{-1}(C_{i_1}) \cap \dots \cap f^{-(n-1)}(C_{i_{n-1}})$.

Lemma 6.2. *If the second condition of the proposition does not hold, then there exists $n \geq 2$ satisfying the following. For any $x \in \Lambda$ there exists a sub rectangle $C' \subset C(x)$ in $\mathcal{C}^n$ such that $W^{uu}_{loc}(x) \cap C' = \emptyset$.*

Proof. Assume that the second condition of the proposition does not hold: there exist two points z_1, z_2 in the same unstable manifold such that $W^{uu}(z_1)$ and $W^{uu}(z_2)$ are different. Taking a negative iterate if necessary, we may assume that z_1, z_2 belong to the same local unstable manifold and the same rectangle C. In particular, there exists $m > 1$ and two subrectangles $C_1, C_2 \in \mathcal{C}^m$ with $z_1 \in C_1$ and $z_2 \in C_2$ such that the plaque $W^u_{loc}(z_1)$ satisfies the following property: for any

$x_1 \in C_1 \cap W^u_{loc}(z_1)$ and $x_2 \in C_2 \cap W^u_{loc}(z_1)$, the manifolds $W^{uu}_{loc}(x_1)$ and $W^{uu}_{loc}(x_2)$ are disjoint. By compactness, the local unstable manifold of any point z close to z_1 satisfies the same property.

Let us consider any point $x \in \Lambda$. Since Λ is locally maximal and transitive, there exists a point z close to z_1 having a backward iterate $f^{-p}(z)$ in $W^u_{loc}(x) \cap C(x)$. It follows that $C(x)$ contains two rectangles $C'_1, C'_2 \in \mathcal{C}^{m+p}$ satisfying: for any $x_1 \in C'_1 \cap W^u_{loc}(x)$ and $x_2 \in C'_2 \cap W^u_{loc}(x)$, the manifolds $W^{uu}_{loc}(x_1)$ and $W^{uu}_{loc}(x_2)$ are disjoint. In particular, $W^{uu}_{loc}(x)$ is disjoint from C'_1 or C'_2 and the lemma holds for the point x and any integer n larger than $m+p$. Note that it also holds for any point $x' \in \Lambda$ close to x. By compactness we obtain that there exists $n \geq 1$ such that the lemma holds for all $x \in \Lambda$. $\square$

We now continue with the proof of the proposition. We assume that the second case does not hold and consider an integer n as in the previous lemma. For any $x \in \Lambda$ and $m \geq 1$ we denote by $T_m(x)$ the union of the rectangles in $\mathcal{C}^m$ that are contained in $C(x)$ and meet $W^{uu}_{loc}(x)$.

Since μ has full support Λ and since its disintegration is invariant under stable holonomy, there exists $\tau > 0$ such that for any $x \in \Lambda$ and any $C' \in \mathcal{C}^n$ contained in $C(x)$, we have

$$\mu^u(C' \cap W^u_{loc}(x)) > \theta.\mu^u(C(x) \cap W^u_{loc}(x)).$$

It follows that

$$\mu^u(T_n(x) \cap W^u_{loc}(x)) < (1-\theta).\mu^u(C(x) \cap W^u_{loc}(x)).$$

Since μ^u has constant jacobians along the unstable leaves, we have

$$\mu^u(T_{(j+1).n}(x) \cap W^u_{loc}(x)) < (1-\theta).\mu^u(T_{j.n}(x) \cap W^u_{loc}(x)).$$

The measure $\mu^u(W^{uu}_{loc}(x) \cap C(x))$ is the limit of $\mu^u(T_{j.n}(x) \cap W^u_{loc}(x))$ which is exponentially small. Thus $\mu^u_{loc}(W^{uu}(x))$ has zero μ^u-measure, for all $x \in \Lambda$. $\square$

6.2. **A reverse doubling property of partially hyperbolic horseshoes.**

In this subsection and the following ones, we will consider a diffeomorphism f and a horseshoe Λ satisfying the following hypothesis:

(H) f is a $C^{1+\alpha}$-diffeomorphism for some $\alpha > 0$ and the horseshoe Λ has a partially hyperbolic splitting:

$$T_\Lambda M = E^{uu} \oplus E^c \oplus E^s,$$

where E^s and $E^u = E^{uu} \oplus E^c$ are the stable and unstable bundles in the hyperbolic splitting for $f|_\Lambda$ and where E^c is one-dimensional and essential.

6.2.1. *The reverse doubling property.* We prove a geometric inequality of independent interest for the disintegration of the measure of maximal entropy along unstable leaves. A measure satisfying this inequality is sometimes said to have the *reverse doubling property* – for a discussion of this property, see [HMY]. We will later need a more technical version of this property (see Lemma 6.10) that will be proved analogously.

Theorem 6.3 (Reverse doubling property)**.** *For any diffeomorphism f and any horseshoe Λ satisfying the property (H), there exist $\rho, \eta > 0$ such that for any $x \in \Lambda$ and $r \in (0, \rho)$,*

$$\mu^u \left(W^u(x, \eta r)\right) < \frac{1}{2}\mu^u \left(W^u(x, r)\right). \tag{13}$$

We will at times be interested in replacing Λ by subhorseshoes that have a large period, i.e. that have a partition $K \cup f(K) \cup \dots f^{N-1}(K)$ into disjoint compact sets for some large integer N. The next result states that these subhorseshoes can be extracted to have large entropy and a uniform reverse doubling property.

Theorem 6.4 (Reverse doubling property and large period)**.** *For any diffeomorphism f and any horseshoe Λ satisfying the property (H) and for any $\varepsilon > 0$, there exist $\eta, \rho > 0$ with the following property.*

For any $N_0 \geq 1$ there exist $N \geq N_0$ and a subhorseshoe $\Lambda_N \subset \Lambda$ that:

- *admits a partition into compact subsets $\Lambda_N = K \cup \dots f^{N-1}(K)$,*
- *has entropy larger than $h_{top}(\Lambda, f) - \varepsilon$,*
- *for any $x \in \Lambda_N$ and $r \in (0, \rho)$, we have:*

$$\mu^u \left(W^u(x, \eta r)\right) < \frac{1}{2}\mu^u \left(W^u(x, r)\right).$$

6.2.2. *Split Markov partitions.* The reverse doubling property will be obtained from a special construction of Markov partitions satisfying a geometrical property that we introduce now. This construction uses strongly that the unstable bundle splits as a sum $E^u = E^{uu} \oplus E^c$ with $\dim(E^c) = 1$.

Let us fix $\rho_0 > 0$ small. Since Λ has a local product structure, for any $x, y \in \Lambda$ close, the intersection $[x, y] := W^u(x, \rho_0) \cap W^s(y, \rho_0)$ is transverse, and consists of a single point that belongs to Λ. A *rectangle* of Λ is a closed and open subset with diameter smaller than ρ_0 that is saturated by the local product: for any x, y in a rectangle, $[x, y]$ also belongs to the rectangle.

Definition 6.5. A *split rectangle* of Λ is a rectangle R that can be expressed as the disjoint union of two subrectangles R_-, R_+ such that (see Figure 5):

- the decomposition is saturated by local stable manifolds: any two points $x, y \in R$ with $y \in W^s(x, \rho_0)$ belong to a same subrectangle R_- or R_+; and
- inside unstable leaves, the subrectangles R^-, R^+ are bounded by strong unstable leaves: for any $x \in R$, there exists $z \in W^u(x, \rho_0)$ such that $R^- \cap W^u(x, \rho_0)$ and $R^+ \cap W^u(x, \rho_0)$ are contained in two different connected components of $W^u(x, \rho_0) \setminus W^{uu}(z, 2\rho_0)$.

A *split Markov partition* of Λ is a collection of pairwise disjoint split rectangles $\{R_i = R_{i,-} \cup R_{i,+},\ i = 1, \dots, m\}$ such that both $\{R_i\}$ and $\{R_{i,-}\} \cup \{R_{i,+}\}$ are Markov partitions.

The next result says that one can extract subhorseshoes with large period and with a split Markov partition.

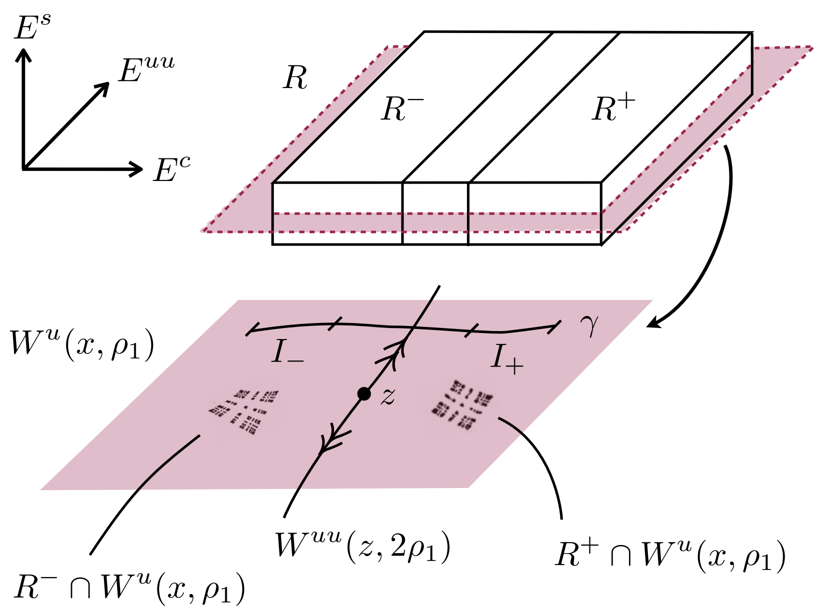

FIGURE 5. A split rectangle and its intersection with an unstable plaque.

Proposition 6.6. *For any diffeomorphism f and any horseshoe Λ satisfying the property (H), for any $\rho', \varepsilon > 0$, there exists a horseshoe $\Lambda' \subset \Lambda$ endowed with a split Markov partition $\{R_{1,\pm}, \dots, R_{m,\pm}\}$ whose rectangles have diameter smaller than ρ' and such that the following property holds.*

For every $N_0 \geq 0$, there exist $N \geq N_0$ and a horseshoe $\Lambda_N \subset \Lambda'$ such that:
- *Λ_N admits a decomposition into compact subsets*
$$\Lambda_N = K \cup f(K) \cup \dots \cup f^{N-1}(K),$$
- *the topological entropy $h_{top}(\Lambda_N, f)$ is larger than $h_{top}(\Lambda, f) - \varepsilon$,*
- *for each i and $x \in \Lambda_N \cap R_i$, the disintegration μ^u of the measure of maximal entropy of Λ_N along the unstable leaves gives the same weight to $W^u(x, \rho_0) \cap R_{i,+}$ and $W^u(x, \rho_0) \cap R_{i,-}$.*

Proof. Consider a Markov partition $C_0, \dots, C_s$ of Λ into disjoint compact rectangles with diameter smaller than ρ'. We can furthermore require that the maximal invariant set A in the smaller collection $C_1 \cup \dots \cup C_s$ has entropy larger than $h_{top}(\Lambda, f) - \varepsilon/4$. A word $i_1, \dots, i_n$ in $\{0, \dots, s\}^n$ is *admissible* if $f^{n-1}(C_{i_1}) \cap f^{n-2}(C_{i_2}) \cap \dots \cap C_{i_n} \neq \emptyset$.

Fix some point $x_0 \in C_0$. Since E^c is essential, there exist $x^-, x^+ \in C_0 \cap W^u(x_0, \rho_0)$ such that $W^{uu}(x^{\pm}, 2\rho_0)$ are disjoint. For $\ell \geq 1$ let
$$(i_{-\ell}, \dots, i_{-1}, 0, i_1^-, \dots, i_\ell^-),\ (i_{-\ell}, \dots, i_{-1}, 0, i_1^+, \dots, i_\ell^+)\ \in \{0, \dots, s\}^{2\ell+1}$$
denote the itinerary of $f^{-\ell}(x^+), \dots, f^\ell(x^+)$ and $f^{-\ell}(x^-), \dots, f^\ell(x^-)$ in the partition $C_0, \dots, C_s$. If ℓ is large enough, the rectangles
$$R_- = f^\ell(C_{i_{-\ell}}) \cap \dots \cap C_0 \cap f^{-1}(C_{i_1^-}) \cap f^{-\ell}(C_{i_\ell^-}),$$

$$R_+ = f^\ell(C_{i_{-\ell}}) \cap \cdots \cap C_0 \cap f^{-1}(C_{i_1^+}) \cap f^{-\ell}(C_{i_\ell^+}),$$

are small neighborhoods of x^-, x^+ in Λ. The union $R = R_- \cup R_+$ is a split rectangle. Note that one can modify x^-, x^+ and take ℓ so that $i_\ell^- = i_\ell^+$.

We then consider some integer L large and the admissible words of length L of the form $i_1^-, \dots, i_\ell^-, w, i_{-\ell}, \dots, i_{-1}, 0$ or $i_1^+, \dots, i_\ell^+, w, i_{-\ell}, \dots, i_{-1}, 0$ with admissible words w in $\{1, \dots, s\}$ of length $L - 2\ell - 1$. The bi-infinite words obtained by concatenation of these words define the horseshoe Λ'. There exist L arbitrarily large such that the entropy of Λ' is arbitrarily close to the entropy of A, hence is larger than $h_{top}(\Lambda, f) - \varepsilon/3$.

If L has been chosen large enough, we note that $R \cap \Lambda'$ is disjoint from its $L-1$ first iterates. Hence, any segment of orbit of Λ' of length L meets R at one point exactly. One can thus define a Markov partition by taking rectangles of the form

$$f^{-n}(R \cap \Lambda') \cap f^{-n+1}(C_{j_1}) \cap \cdots \cap f^{-1}(C_{j_{n-1}}) \cap C_{j_n},$$

with n varying between 0 and L. These rectangles are compact, disjoint and naturally split by $f^{-n}(R_\pm)$. We thus get a collection of disjoint split rectangles $R_{1,\pm}, \dots, R_{m,\pm}$ of Λ'. Both partitions $\{R_i\}$ and $\{R_{i,-}\} \cup \{R_{i,+}\}$ are Markov by construction.

Since $i_\ell^- = i_\ell^+$, the set of itineraries of the forward orbits from R_- and R_+ are equal after time ℓ. This implies that the disintegration of the maximal entropy measure of Λ' gives the same weight to $W^u(x, \rho_0) \cap R_+$ and $W^u(x, \rho_0) \cap R_-$ for each $x \in R \cap \Lambda'$. Similarly, it gives the same weight to $W^u(x, \rho_0) \cap R_{i,+}$ and $W^u(x, \rho_0) \cap R_{,i-}$ for each $x \in R_i$.

Fix a Markov rectangle R' that meets R after $L - \ell$ iterates: if L is large enough, the set of points of Λ' whose orbit does not intersect R' has entropy larger than $h_{top}(\Lambda, f) - \varepsilon/2$. For N large (and a multiple of L), the subhorseshoe Λ_N corresponds to itineraries that visit R' exactly every N-iterates. Since N is large, the entropy of Λ_N is larger than $h_{top}(\Lambda, f) - \varepsilon$. This horseshoe is N-periodic: setting $K := R' \cap \Lambda_N$, it can be decomposed as

$$\Lambda_N = K \cup f(K) \cup \cdots \cup f^{N-1}(K).$$

Again by symmetry of the construction the disintegration of the maximal entropy measure on Λ' gives the same weight to $W^u(x, \rho') \cap R_{i,+}$ and $W^u(x, \rho') \cap R_{i,-}$ for each i and each $x \in R_i \cap \Lambda_N$. This gives the proposition. $\square$

6.2.3. *Proof of the reverse doubling property.* We now prove Theorem 6.4 (the proof of Theorem 6.3 is similar).

Proof of Theorem 6.4. Since there is a dominated splitting between E^{uu} and E^c, there exists a thin cone field $\mathcal{C}^c$ defined on a small neighborhood of Λ and containing the bundle E^c such that for any $x \in \Lambda$, and any small C^1-curve γ in $W^u(x)$ containing x and tangent to $\mathcal{C}^c$, the backward iterates $f^{-n}(\gamma)$ are still tangent to $\mathcal{C}^c$. By a classical distortion argument, since γ is uniformly contracted by backward iterations, and since f is $C^{1+\alpha}$, there exists some uniform constant $C > 0$ such that for any unit vectors v, v' tangent to γ at points $y, y' \in \gamma$ we have

$$C^{-1}\|Df^{-n}(y')(v')\| \le \|Df^{-n}(y)(v)\| \le C\|Df^{-n}(y')(v')\|.$$

Let's apply Proposition 6.6 to f, Λ, ε and a small constant ρ': we obtain a split Markov partition $\{R_{i,\pm}\}$ of $\Lambda' \subset \Lambda$. For each R_i and each $x \in R_i$, there exists a small curve $\gamma \subset W^u(x, \rho_0)$ tangent to $\mathcal{C}^c$ and two disjoint compact intervals $I_+, I_- \subset \gamma$ such that for any $y \in R_i \cap W^u(x, \rho_0)$ the strong manifold $W^{uu}(y, \rho_0)$ intersects γ at a point of I_+ if $y \in R_{i,+}$ and at a point of I_- otherwise. (See Figure 5.) If ρ' is small, the curve γ is also small.

There exist two constants $\widehat{L}, L > 0$, which do not depend on $x \in \Lambda'$ or on $\gamma \in W^u(x, \rho_0)$, such that:

- the length of γ is smaller than $\widehat{L}$,
- the distance between I_- and I_+ in γ is larger than L.

Choose constants $\rho, \eta > 0$ small. For any a horseshoe Λ_N given by Proposition 6.6, we have to prove the reverse doubling property for these constants.

We fix $x \in \Lambda_N$ and $r \in (0, \rho)$. For each $\zeta \in B(x, \eta r) \cap W^u(x, \rho_0) \cap \Lambda'$ we consider the sequence (i_k) such that $f^k(\zeta) \in R_{i_k}$ for each $k \geq 0$ and then define the domain

$$\Delta_{\zeta,k} := f^{-k}(R_{i_k}) \cap f^{-k+1}(R_{i_{k-1}}) \cap \cdots \cap R_{i_0} \cap W^u(x, \rho_0);$$

it splits in two pieces $\Delta_{\zeta,k,\pm} := \Delta_{\zeta,k} \cap f^{-k}(R_{i_k,\pm})$. We also consider Δ_ζ, the largest domain $\Delta_{\zeta,k}$ contained in $B(x, r)$, and $\Delta_{\zeta,\pm}$ the corresponding domains $\Delta_{\zeta,k,\pm}$. Note that if ρ has been chosen small, then k is large.

Next choose γ, I_-, I_+ associated to R_{i_k} and $W^u(f^k(\zeta), \rho_0)$ as above. From the domination between E^{uu} and E^c, the domains $\Delta_{\zeta,k}$ and $\Delta_{\zeta,k,\pm}$ are contained in the unions

$$\Delta_{\zeta,k} \subset \bigcup_{z \in \gamma} f^{-k}(W^{uu}(z, \rho_0)), \quad \Delta_{\zeta,k,\pm} \subset \bigcup_{z \in I_\pm} f^{-k}(W^{uu}(z, \rho_0)). \tag{14}$$

The diameter of the strong unstable disk $f^{-k}(W^{uu}(z, \rho_0))$ is much smaller than the length of the curves $f^{-k}(\gamma)$, $f^{-k}(I_\pm)$. Hence the domains $\Delta_{\zeta,k}$, $\Delta_{\zeta,k,\pm}$ are strips that are very thin in the strong unstable direction.

From the distortion estimate, the distance between $f^{-k}(I^-)$ and $f^{-k}(I^+)$ is larger than

$$C^{-2}\frac{L}{\widehat{L}}\text{Length}(f^{-k}(\gamma)).$$

The holonomy along the strong-unstable foliation between central curves inside the unstable plaques is uniformly Lipschitz. As a consequence there exists a uniform constant $\kappa > 0$ such that if γ_k, γ_{k+1} are central curves associated to the domains $\Delta_{\zeta,k}$ and $\Delta_{\zeta,k+1}$, then we have

$$\text{Length}(f^{-(k+1)}(\gamma_{k+1})) \geq (2\kappa)\text{Length}(f^{-k}(\gamma_k)). \tag{15}$$

By maximality of $\Delta_\zeta \subset B(x, r)$, the length of the associated curve $f^{-k}(\gamma)$ is thus larger than κr. It follows that the distance between $f^{-k}(I^-)$ and $f^{-k}(I^+)$ is larger than $C^{-2}\kappa r L/\widehat{L}$.

If η is chosen smaller than $C^{-2}\kappa L/\widehat{L}$, then only one domain $\Delta_{\zeta,-}$ or $\Delta_{\zeta,+}$ can intersect $B(x, \eta r)$. Since f has constant Jacobian along the unstable leaves for the measure μ^u, both preimages have the same μ^u-measure. This proves that

$$\mu^u(\Delta_\zeta \cap B(x, \eta r)) \leq \frac{1}{2}\mu^u(\Delta_\zeta).$$

By construction the domains Δ_ζ for $\zeta \in B(x, \eta r) \cap W^u(x, \rho_0) \cap \Lambda'$ are disjoint or equal. We thus obtain a finite family Y such that the domain Δ_ζ, $\zeta \in Y$ are pairwise disjoint, and their union contains $B(x,\eta r)\cap W^u(x,\eta r)\cap\Lambda'$ and is contained in $B(x,r)$. This proves that

$$\mu^u(B(x,\eta r)\cap W^u(x,\rho_0)) = \sum_{\zeta\in Y}\mu^u(\Delta_\zeta\cap B(x,\eta r))$$
$$\leq \sum_{\zeta\in Y}\frac{1}{2}\mu^u(\Delta_\zeta)\leq \frac{1}{2}\mu^u(B(x,r)\cap W^u(x,\rho_0)).$$

This gives the desired estimate. □

6.3. **Extraction of sparse horseshoes and proof of Theorem 1.3.** We first prove the existence of a subhorseshoe with large entropy and nice geometry.

Theorem 6.7 (Sparse subhorseshoe)**.** *For any diffeomorphism f, any horseshoe Λ satisfying (H), and any $\varepsilon>0$, there exist $\chi,\lambda\in(0,1)$ with the following property.*

For any $\rho>0$ there exist compact subsets $X_1,\dots,X_n\subset\Lambda$ such that

- *each X_i has diameter in $[\lambda\rho,\rho]$;*
- *the neighborhoods $\widehat{X}_i$ of size $\chi.\mathrm{diam}(X_i)$ of the X_i are pairwise disjoint;*
- *the entropy of the restriction of f to the maximal invariant set in*

$$\bigcup_{i=1}^{n}\bigcup_{j=0}^{N-1} f^j(X_i)$$

is larger than $h_{top}(\Lambda,f)-\varepsilon$.

The proof of this theorem is postponed until Section 6.5

Proof of Theorem 1.3 from Theorem 6.7. Theorem 1.3 will be proved in several steps: we will first show that Λ admits an extracted horseshoe that can be locally linearized; we then show that this subhorseshoe admits a dominated decomposition into one-dimensional subbundles and a global chart where these bundles are constants. At last we prove that after another extraction there is a subhorseshoe which is globally linear.

Fix $\varepsilon>0$. Note that by Theorem 1.4, we can perform if necessary an arbitrarily small C^r-perturbation and replace Λ by a subhorseshoe whose topological entropy is arbitrarily close to the initial entropy in order to ensure that Λ has a dominated splitting $T_\Lambda M = E_1\oplus\cdots\oplus E_d$ into one dimensional subbundles.

One can by a small C^1-perturbation replace f by a diffeomorphism that is C^∞ in a small neighborhood of Λ. This is more delicate in the conservative setting: one uses [Z] for symplectic diffeomorphisms, [DM] for volume preserving ones in the case $r>1$ and [Av, Theorem 5] for volume preserving ones in the case $r=1$.

By an arbitrarily C^r-small perturbation we can also assume that E^c is essential, so that the property (H) holds. Indeed if p,q are two periodic points close in Λ, it is possible to break the intersection between the local stable manifold of p and the local strong unstable manifold of q (tangent to $E^{uu}(q)$); the intersection between $W^u_{loc}(q)$ and $W^s_{loc}(p)$ belongs to $\Lambda\cap(W^u(q)\setminus W^{uu}(q))$, proving that $\Lambda\cap W^u(q)$ is not contained in $W^{uu}(q)$.

Now apply Theorem 6.7, using $\varepsilon/4$: we obtain $\chi\in(0,1)$. We also fix a global chart $\varphi\colon U\to\mathbb{R}^d$ from a neighborhood U of Λ, so that we can work in $\mathbb{R}^d$; if one considers a volume or a symplectic form, one can take it to be constant in the chart. For $\rho > 0$ that will be chosen small enough, we get a family of compact sets $X_1, \dots, X_n \subset K$ with diameter in $[\lambda.\rho, \rho]$ and we introduce for each X_i its $\chi.\mathrm{diam}(X_i)/2$-neighborhood Q_i and its $\chi.\mathrm{diam}(X_i)$- neighborhood $\widehat{Q}_i$. The sets $\widehat{Q}_i$ are pairwise disjoint, have a diameter smaller than $(1+2\chi).\rho$ and there exists $\chi' > 0$ (which depend on f, χ, λ, but not on ρ) such that $\widehat{Q}_i$ contains the $\chi'.\mathrm{diam}(Q_i)$-neighborhood of Q_i.

For each i, we choose a point $z_i \in Q_i$ and a linear map A_i in $GL(\mathbb{R}, d)$ close to $Df(x)$ for any $x \in \widehat{Q}_i$. For instance, one could use $A_i = Df(z_i)$. We will now perturb f in the sets $\widehat{Q}_i$: we build a diffeomorphism g that coincides with the affine map $y \mapsto A_i \cdot (y - z_i) + f(z_i)$ on Q_i. This is done with the following lemma.

Lemma 6.8 (Local linearization)**.** *For any $d \geq 1$ and any $\chi', \theta > 0$, there exists $\eta > 0$ such that any compact set $Q \subset \mathbb{R}^d$ satisfies the following property.*

For any $B \in GL(\mathbb{R}, d)$ such that $\|B - \mathrm{Id}\| < \eta$ and any $y_0 \in Q$, there exists a C^∞-diffeomorphism H that coincides with the identity outside the $\chi'.\mathrm{diam}(Q)$-neighborhood of Q, that coincides with $y \mapsto B(y - y_0) + y_0$ in Q and that satisfies $\|DH - \mathrm{Id}\| \leq \theta$ everywhere.

If B preserves a given volume or symplectic form, then H can be chosen to preserve it as well.

Proof. Fix $\theta, \chi' > 0$ and $d \geq 1$. We note that it is enough to prove the result for compact sets Q containing 0 and whose diameter equals 1.

Let us first fix a compact set $Q_0 \subset \mathbb{R}^d$ containing 0 with $\mathrm{diam}(Q_0) = 1$ and prove that there exists $\eta > 0$ satisfying the conclusion of the lemma for Q_0. We choose two disjoint, closed neighborhoods Δ_0, Δ_1 of $\{y : d(y, Q_0) \geq \chi\}$ and Q_0 respectively. We consider a smooth map $\sigma\colon \mathbb{R}^d \to [0,1]$ that coincides with 0 on Δ_0 and with 1 on Δ_1. The smooth map $H = \sigma \cdot (B(y - y_0) + y_0) + (1 - \sigma) \cdot \mathrm{Id}$ is a diffeomorphism and is C^∞-close to the identity (uniformly in $y_0 \in Q_0$), provided that $\|B - \mathrm{Id}\| < \eta$ on $\widehat{Q}_0$ for some small $\eta > 0$.

When B preserves Lebesgue, one obtains a conservative diffeomorphism which coincides with the identity on Δ_0 and with $y \mapsto B(y - y_0) + y_0$ on Δ_1 by applying [DM] (see also [Av, Corollary 4]). The diffeomorphism is C^∞-close to the identity if $\|B - \mathrm{Id}\|$ is small.

When B is symplectic, the map $y \mapsto B(y - y_0) + y_0$ is the time-one map of a flow associated to a Hamiltonian h_B which is close to 0 on Δ_1^c, provided $\|B - \mathrm{Id}\|$ is small. The map H can be directly obtained as the time-one map of the flow associated to the Hamiltonian $\sigma.h_B$ where σ is the same map as above.

Note that the triple $(\Delta_0, \Delta_1, \eta)$ is still valid for any compact set Q that is Hausdorff close to Q_0. Since the set of compact sets whose diameter is equal to 1 and which contain 0 is compact in the Hausdorff topology, there exists $\eta > 0$ which is valid for all compact sets. □

Applying the previous lemma provides us with a diffeomorphism h which is C^1-close to the identity and supported in the sets $\widehat{Q}_i$. We then define $g = h \circ f$. The tangent maps of f and g are C^0-close, and since the size of the connected components of the support of the perturbation is smaller than $\chi.\rho$, the C^0 distance between f and g is also small. Consequently g belongs to the neighborhood $\mathcal{U}$ of f. Moreover if f preserves a volume or a symplectic form, by choosing the linear maps

A_i to be conservative, the perturbation g still preserves the form and is locally affine in the union $V := \bigcup_i Q_i$.

Consider a transitive hyperbolic set Λ' with entropy larger than $h_{top}(\Lambda, f) - \varepsilon/4$ and contained in $\bigcup_i X_i$ as given by Theorem 6.7. Since f and g are $(1+2\chi).\rho$-close in the C^0 topology, the shadowing lemma implies that the hyperbolic continuation Λ'_g of Λ' for g is contained in the $\gamma\rho$-neighborhood of Λ', where γ is arbitrarily close to 0 if the distance between Df and Dg is chosen small enough. In particular, Λ'_g is contained in the union V, hence is locally affine. Moreover Λ'_g has entropy larger than $h_{top}(\Lambda, f) - \varepsilon/4$. By Theorem 1.5, there exists a horseshoe $\widetilde{\Lambda}$ with entropy larger than $h_{top}(\Lambda, f) - \varepsilon/2$ contained in V, hence locally affine as required. This gives the first part of Theorem 1.3.

We now explain how to modify the previous construction so that the splitting $T_{\widetilde{\Lambda}}M = E_1 \oplus \cdots \oplus E_d$ is locally constant. We first introduce a family of disjoint open sets $\Delta_1, \dots, \Delta_s$ which cover Λ and have small diameters. We choose a point $x_k \in \Delta_k \cap \Lambda$ in each of them. The sets $X_1, \dots, X_n$ given by Theorem 6.7 are only chosen after, with diameter small enough so that each set Q_i and each image $f(Q_i)$ is contained in one of the sets Δ_k. We linearize in each domain as before but require that g coincide on Q_i with the affine map $z \mapsto A_i.(z - z_i) + f(z_i)$ where $A_i = \Pi_{i,,k,\ell} \circ Df(x_k)$, such that

– k and ℓ are defined by the conditions $Q_i \subset \Delta_k$ and $f(Q_i) \subset \Delta_\ell$,
– the linear maps $\Pi_{i,k,\ell}$ is close to the identity and sends the splitting $E_1(x_k) \oplus \cdots \oplus E_d(x_k)$ to the splitting $E_1(x_\ell) \oplus \cdots \oplus E_d(x_\ell)$.

When f preserves the volume or the symplectic form, we choose $\Pi_{i,k,\ell}$ to preserve it as well. In the symplectic case this is possible since the two planes of the form $E_m \oplus E_{d-m}$ are pairwise symplectic-orthogonal (see [BV1]).

The end of the construction is unchanged. After perturbation, the dominated splitting on $\widetilde{\Lambda}$ for the map g, coincides in each set Δ_k with $E_1(x_k) \oplus \cdots \oplus E_d(x_k)$ and hence is locally constant.

At this step we have reduced the proof of Theorem 1.3 to the case of a local diffeomorphism f of a hyperbolic set Λ in a subset U of $\mathbb{R}^d$, such that f is locally affine on U, and the splitting of $\mathbb{R}^d$ into the coordinates axes is f-invariant. In the conservative case, the volume, or the symplectic form is chosen to coincide with the standard Lebesgue volume of $\mathbb{R}^d$ or with the standard symplectic form of $\mathbb{R}^{2\times\frac{d}{2}}$. It remains to prove that after a new extraction and a new perturbation, the linear part can be made constant.

Choose a Markov partition of Λ into small disjoint rectangles $R_0, \dots, R_\ell$. Since the R_i are small, the diffeomorphism f is affine on a neighborhood of R_i: there exists a linear map A_i such that the diffeomorphism f has the form $z \mapsto A_i.(z-x)+f(x)$. Since the coordinates axes are preserved, A_i is diagonal with diagonal coefficients $a_{i,1}, \dots, a_{i,d}$.

For $N \geq 1$ large enough, consider the sub-horseshoe Λ_N of points of Λ which visit R_0 exactly every N iterates: its topological entropy is larger than $h_{top}(\Lambda, f) - \varepsilon/4$. Moreover the return map f^N on $R_0 \cap \Lambda_N$ decomposes into branches labelled by compatible itineraries in $R_1, \dots, R_\ell$, which are affine. Each itinerary is associated to a diagonal matrix whose coefficients are sums of N numbers chosen among $a_{i,k}$, with $0 \leq i \leq \ell$ and $1 \leq k \leq d$. The number of such matrices grows at most polynomially in N. On the other hand, the number of itineraries of length N starting in $R_0 \cap \Lambda_N$ grows faster than $e^{N(h_{top}(\Lambda,f)-\varepsilon/4)}$.

It follows from the pigeonhole principle that there exists a diagonal matrix A and a set containing at least $e^{N(h_{top}(\Lambda,f)-\varepsilon/2)}$ itineraries of length N which start in $R_0 \cap \Lambda_N$ and which all have the same associated diagonal matrix A^N. The set of points in Λ_N whose orbit follows these itineraries is a sub horseshoe $\widetilde{\Lambda}$ whose topological entropy is larger than $h_{top}(\Lambda, f) - \varepsilon/2$ as required. It may be decomposed as a disjoint union of compact sets: $\widetilde{\Lambda} = K \cup f(K) \cup \dots \cup f^{N-1}(K)$. The diffeomorphism f^N on K is affine with the constant linear part A^N.

Now cover K by small disjoint open sets $\Delta_1, \dots, \Delta_m$ is such a way that f^ℓ is affine on each Δ_j, with $0 \le \ell \le N$ and $1 \le j \le m$, and has a linear part $B_{j,\ell}$. We change the chart on each set $f^\ell \Delta_j$ by the composition of $A^\ell B_{j,\ell}^{-1}$ with a translation. For this new chart, the diffeomorphism f has the form $z \mapsto A.(z-x) + f(x)$ near each point $x \in \widetilde{\Lambda}$. This completes the proof of Theorem 1.3. $\square$

6.4. **Cube families.** In this subsection and the following ones we prove Theorem 6.7. We thus consider a diffeomorphism f and a horseshoe Λ satisfying (H). We describe some preliminary constructions.

6.4.1. *The affine chart φ - the scale ρ_0 - the metric on Λ.* For each point $x \in \Lambda$, we will use smooth charts ϕ_x from a neighborhood U_x of x in M to $\mathbb{R}^d$ whose derivative $D_x\phi_i$ at x sends the splitting $E^{uu}(x) \oplus E^c(x) \oplus E^s(x)$ to the splitting

$$\mathbb{R}^d = \mathbb{R}^{d_u-1} \oplus \mathbb{R} \oplus \mathbb{R}^{d_s}. \tag{16}$$

Since the angle between the spaces E^{uu}, E^c, E^s is bounded away from zero, one can assume that the C^1-norm of the charts ϕ_x is uniformly bounded by some $D > 0$.

In the case f preserves a volume m or a symplectic form ω, we may assume that $\phi_* m$ (resp. $\phi_*\omega$) is the standard symplectic form on $\mathbb{R}^{2\times\frac{d}{2}}$. Indeed, for the volume preserving case one uses [DM]. In the symplectic case, we first use Darboux's theorem in order to rectify the symplectic form. By [BV1], the horseshoe Λ admits a finer dominated splitting into Lagrangian subbundles

$$T_\Lambda = E^{uu} \oplus E^c_1 \oplus E^c_2 \oplus E^{ss}$$

such that $E^c = E^c_1$, $E^s = E^c_2 \oplus E^{ss}$, $\dim(E^{uu}) = \dim(E^{ss})$ and $\dim(E^c_1) = \dim(E^c_2)$. The subspaces $E^{uu} \oplus E^{ss}$ and $E^c_1 \oplus E^c_2$ are symplectic orthogonal. All this implies that the splitting $T_xM = E^{uu}(x) \oplus E^c(x) \oplus E^s(x)$ may be sent by a bounded symplectic linear map to the standard splitting (16).

Since Λ is totally disconnected, for any scale $\rho_0 > 0$ there exist a neighborhood U of Λ having finitely many connected components U_k, each of them has diameter smaller than ρ_0, contains a point $x_k \in \Lambda$, and is included in the domain of the chart ϕ_{x_k}. We may also assume that the images $\phi_{x_k}(U_k)$ are pairwise disjoint, so that the map $\varphi\colon U \to \mathbb{R}^d$ which coincides with ϕ_{x_k} on U_k is also a chart.

Two metrics appear: the initial metric on M and the chart metric induced by φ from the standard metric in $\mathbb{R}^d$. At this stage, the chart φ depends on ρ_0 and has not been fixed. However the initial and charts metrics are comparable up to the constant D which does not depend on φ. In the following we will mainly consider the chart metric.

6.4.2. *First extraction - the Lyapunov exponents - the split Markov partition.* Replacing if necessary Λ by a subhorseshoe whose entropy is arbitrarily close to the entropy of the initial horseshoe, one can assume that:

– (Theorem 1.5.) There exist constants $-\lambda \le -\nu < 0 < \gamma < \widehat{\gamma} < \widehat{\nu} \le \widehat{\lambda}$ such that for any invariant probability measure on Λ the Lyapunov exponents along E^{uu}, E^c, and E^s belong to $(\widehat{\nu}, \widehat{\lambda})$, $(\gamma, \widehat{\gamma})$ and $(-\lambda, -\nu)$ respectively.

  One chooses N_0 large enough so that for any $N \ge N_0$, if u is a unit vector in E^{uu}, E^c or E^s, then $\frac{1}{N}\log\|Df^N(u)\|$ belongs to the corresponding interval.

– (Proposition 6.6.) For any $\varepsilon' > 0$, there are subhorseshoes that are disjoint unions of the form $\Lambda_N = K \cup \dots f^{N-1}(K)$ with $N \ge N_0$ and entropy larger than $h_{top}(\Lambda, f) - \varepsilon'$, and there exists a split Markov partition $\{R_{1,\pm}, \dots R_{m,\pm}\}$ whose rectangles have a small diameter and such that μ^u gives the same weight to $R_{i,-}$ and $R_{i,+}$ inside each unstable plaque.

6.4.3. *Sheared cubes - the shear σ - the scale 2^{-k}.* Let $\{e_1, \dots, e_{(d_u-1)}\}$, $\{e_{d_u}\}$, $\{f_1, \dots, f_{d_s}\}$ be orthonormal bases for the three factors of the decomposition (16). For a shear $\sigma \in (0,1)$ we define the linear transformation $L_\sigma \colon \mathbb{R}^d \to \mathbb{R}^d$ by $L_\sigma(e_j) = e_j$ if $1 \le j < d_u$, $L_\sigma(f_j) = f_j$ if $1 \le j \le d_s$ and

$$L_\sigma(e_{d_u}) = \sum_{j=1}^{d_u} e_j + \sigma \cdot \sum_{j=1}^{d_s} f_j.$$

We denote by P_σ the image $L_\sigma(P)$ of the unit parallelepiped centered at 0:

$$P = \{t_1 e_1 + \dots + t_{d_u} e_{d_u} + t_{(d_u+1)} f_1 + \dots + t_d f_{d_s},\ t_i \in [-1/2, 1/2]\}.$$

The reason we need a shear will appear in Lemmas 6.10 and 6.12.

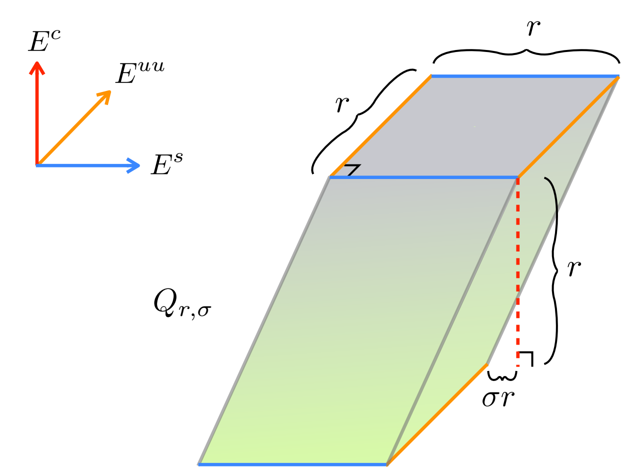

FIGURE 6. The dimensions of a cube $Q_{r,\sigma}(y)$ of diameter r and shear σ.

The *unstable faces* are the faces of P_σ spanned by the same vectors as P_σ, but one among $e_1, \dots, e_{d_u-1}, L_\sigma(e_{d_u})$. The other faces are the *stable faces* and are spanned by the same vectors as P_σ, but one among $f_1, \dots, f_{d_s}$. The *unstable boundary* $\partial^u P_\sigma$ (resp. *stable boundary*) is the union of the unstable (resp. stable) faces.

Fix $\rho_1 \ll \rho_0$. For $r \in (0, \rho_1)$ and $y \in U$ at distance less than ρ_1 from Λ, we define the *(sheared) cube* (see Figure 6)

$$Q_{r,\sigma}(y) = \varphi^{-1}\left(rP_\sigma + \varphi(y)\right) = \varphi^{-1}\left(rL_\sigma P + \varphi(y)\right).$$

The point y is called the *center* of the *cube* $Q_{r,\sigma}(y)$ and r is its *diameter*. For any cube Q with center y and diameter r, and any $\kappa \in (0,2)$, we denote by κQ the cube centered at y of diameter κr. The stable and unstable boundaries of Q are defined analogously to the boundaries of P_σ.

For $\eta \in (0,1)$, we define linear transformations $H^s_\eta, H^u_\eta : \mathbb{R}^d \to \mathbb{R}^d$ by $H^s_\eta(f_i) = (1-\eta) f_i$, $H^s_\eta(e_i) = e_i$, $H^u_\eta(f_i) = f_i$ and $H^u_\eta(e_i) = (1-\eta)e_i$. For $Q = Q_{\rho,\sigma}(y)$, we define the *stable and unstable* η*-boundaries* of Q as follows:

$$\partial^s_\eta Q = Q \setminus \varphi^{-1}\left(rL_\sigma H^s_\eta P + \varphi(y)\right) \text{ and } \partial^u_\eta Q = Q \setminus \varphi^{-1}\left(rL_\sigma H^u_\eta P + \varphi(y)\right).$$

This partitions Q into $(1-\eta)Q$ and $\partial^s_\eta Q \cup \partial^u_\eta Q$. Note also that $\partial^s Q = \bigcap_{\eta>0} \partial^s_\eta Q$.

Each cube $Q = Q_{r,\sigma}(y)$ has a set of *unstable neighbor cubes* $\mathcal{N}^u(Q)$ of cardinality 3^{d_u}. The set $\mathcal{N}^u(Q)$ consists of the cubes of diameter r and shear σ that are produced from Q by a translation by

$$r\, L_\sigma \left(n_1\, e_1 \; + \; \dots \; + \; n_{d_u - 1}\, e_{d_u-1} \; + \; n_{d_u}\, e_{d_u}\right),$$

where each n_i is taken in $\{-1, 0, 1\}$ (see Figure 7).

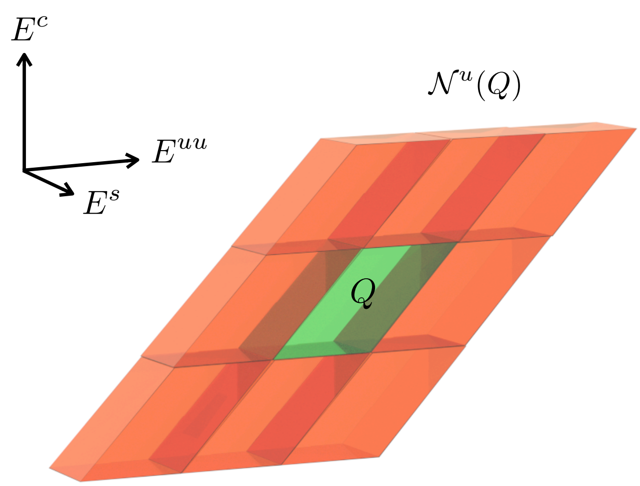

FIGURE 7. The unstable neighbor cubes of Q (which include Q itself).

If ρ_0 is small, then the local unstable manifolds $W^u(z,\rho_0)$ are C^1-close to planes spanned by $e_1, \dots, e_{d_u}$, so that if $(1-3\sigma)Q \cap W^u(z,\rho_0) \neq \emptyset$, then the local unstable manifold $W^u(z,\rho_0)$ does not meet the stable boundary of the $Q' \in \mathcal{N}^u(Q)$, and

$$3Q \cap W^u(z,\rho_0) = \bigcup_{Q' \in \mathcal{N}(Q)} Q' \cap W^u(z,\rho_0).$$

Choose $k > 0$ large. We construct the *cube family* $\mathcal{Q} = \mathcal{Q}_{\varphi,k,\sigma}$ (at scale 2^{-k} with shear σ) as the collection:

$$\mathcal{Q}_{\varphi,k,\sigma} = \left\{Q_{2^{-k},\sigma}(x) : \quad d(x,\Lambda) < \rho_1 \text{ and } \varphi(x) \in 2^{-k}\ L_\sigma(\mathbb{Z}^d)\right\}.$$

6.4.4. *Cube transitions - the boundary size β.* Fix some $\beta > 0$ close to 0. For $Q, Q' \in \mathcal{Q}$, we say that *there is a transition from Q to Q'* (which we denote by $Q \to Q'$) if $f^N(Q)$ intersects Q', whereas $Q' \cap f^N(\partial^u_{\beta/2} Q)$ and $\partial^s_{\beta/2} Q' \cap f^N(Q)$ are empty (see Figure 8).

In the following, we consider the disintegration μ^u of the measure of maximal entropy of a horseshoe Λ_N along the unstable manifolds. We then define the measure μ^u_x induced by μ^u on the plaque $W^u(x, \rho_0)$ for each $x \in K$. We will reduce the proof of Theorem 6.7 to the following proposition, which is proved in Section 6.6.

Proposition 6.9. *Consider f and Λ as introduced in Section 6.4.2.*

For all $\varepsilon > 0$, there exist $\rho_0, \sigma, \beta, N_0 > 0$ and a chart $\varphi\colon U \to \mathbb{R}^d$ with $\Lambda \subset U$, such that if $\Lambda_N \subset \Lambda$ is a subhorseshoe associated to an integer $N \geq N_0$ and if μ^u_x denotes the measure induced on $W^u(x, \rho_0)$ by the disintegration of its measure of maximal entropy along the unstable leaves, then the following holds.

There exists k_0 such that for all $k \geq k_0$, any cube Q in the family $\mathcal{Q} = \mathcal{Q}_{\varphi,k,\sigma}$ of the chart φ and any point $x \in \Lambda_N \cap (1 - 2\beta)Q$, we have:

$$\sum_{Q \to Q'} \mu^u_x \left(f^{-N}((1 - 2\beta)Q')\right) \geq e^{-\varepsilon} \cdot \mu^u_x \left((1 - \beta)Q\right).$$

6.5. **Proof of Theorem 6.7 from Proposition 6.9.** Consider a horseshoe Λ as in the assumptions of Theorem 6.7 and $\varepsilon > 0$. Fix a Markov partition $R_0, \dots, R_s$ of Λ such that any point $x \in \Lambda$ is uniquely determined by its itinerary in the collection of rectangles R_i.

Proposition 6.9 applied with this value of ε provides us with a chart $\varphi\colon U \to \mathbb{R}^d$ satisfying $\Lambda \subset U$, with a boundary size $\beta > 0$ and a shear σ. We also obtain an arbitrarily large integer $N \geq 1$ and a subhorseshoe Λ_N such that $h_{top}(\Lambda_N, f) > h_{top}(\Lambda', f) - \varepsilon/2$ and which decomposes as a disjoint union $\Lambda_N = K \cup f(K) \cup \dots \cup f^{N-1}(K)$.

The initial and chart metrics are equivalent up to a uniform constant. One can thus end the proof with the initial metric on M. We will choose $\chi > 0$ such that for any cube Q in a family $\mathcal{Q}_{\varphi,k,\sigma}$, the $\chi.\mathrm{diam}(Q)$-neighborhood of the cube $(1-\beta/2)Q$ is contained in Q. There exists also $\lambda \in (0, 1)$ such that for any k large and any two cubes $Q, Q' \in \mathcal{Q}_{\varphi,k,\sigma}$, the quantity $10\lambda\mathrm{diam}(Q)$ is bounded by $\mathrm{diam}(Q')$. In this way, for any $\rho > 0$ sufficiently small, we can choose $k \geq 1$ large such that the cubes $Q \in \mathcal{Q} = \mathcal{Q}_{\varphi,k,\sigma}$ have diameter in $[\lambda.\rho, \rho]$. Note also that all points in the same cube have the same itinerary during N iterates with respect to the Markov partition. Let $X_1, \dots, X_n$ be the collection of cubes $(1 - \beta/2)Q$ for $Q \in \mathcal{Q}$ that meet K and let $\widehat{X}_i$ denote the $\chi\rho$-neighborhood. If ρ has been chosen sufficiently small, then the sets $f^\ell(\widehat{X}_j)$ for $0 \leq \ell < N$ and $1 \leq j \leq n$ are pairwise disjoint.

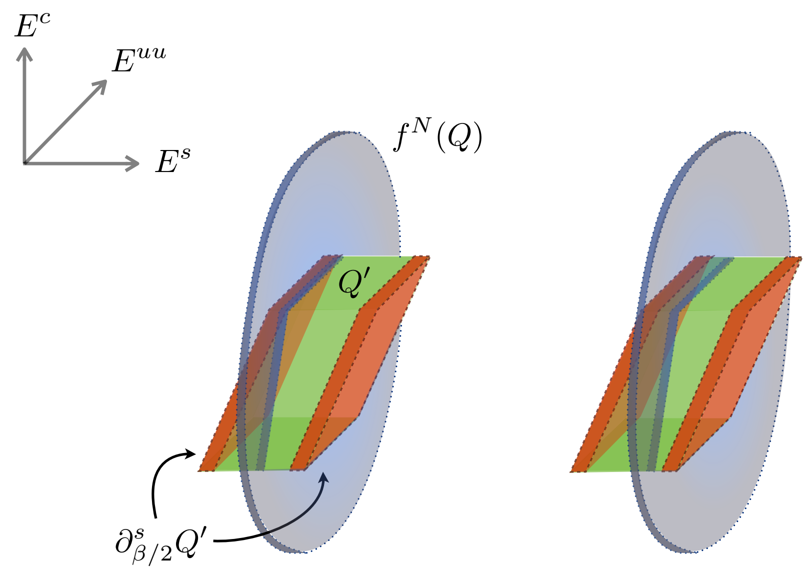

FIGURE 8. An s-bad cube Q' (left) and a transition $Q \to Q'$ (right).

Let V be the union of the cubes $X_1, \dots, X_n$, and let $V_\ell = \bigcap_{j=1}^{\ell} f^{-j.N}(V)$. Fix any point $x \in K$ and $Q \in \mathcal{Q}$ such that $x \in (1-2\beta)Q$. Since μ^u has full support, we have

$$\mu^u_x((1-\beta)Q) > 0.$$

Note that if there is a transition $Q \to Q'$ then for any point $x' \in K$ belonging to $Q' \cap f^N(W^u(x,\rho_0))$ we have $W^u(x',\rho_0) \subset f^N(W^u(x,\rho_0))$ and moreover, the non-empty connected set $f^{-N}(Q') \cap W^u(x,\rho_0)$ is contained in $(1-\beta/2)Q$. Since f has constant Jacobian along the unstable leaves for the measures μ^u, by applying inductively the proposition we obtain that for each $\ell \geq 1$,

$$\sum_{Q \to Q_1 \to \dots \to Q_\ell} \mu^u_x \left(f^{-\ell N}((1-\tfrac{\beta}{2})Q_\ell) \cap f^{-(\ell-1)N}((1-\tfrac{\beta}{2})Q_{(\ell-1)}) \dots \cap (1-\tfrac{\beta}{2})Q \right)$$
$$\geq e^{-\varepsilon \ell} \cdot \mu^u_x\left((1-\beta)Q\right).$$

Hence

$$\mu^u_x\left(V_\ell \cap (1-\beta)Q)\right) > e^{-\ell\varepsilon} \mu^u_x\left((1-\beta)Q\right).$$

Integrating over the different plaques $W^u(x,\rho_0)$ with $x \in (1-2\beta)Q \cap \Lambda_N$, there exists $C_1 > 0$ uniform in ℓ such that the measure of maximal entropy μ on Λ_N satisfies:

$$\mu\left(V_\ell \cap (1-\beta)Q)\right) > C_1 e^{-\ell\varepsilon}. \tag{17}$$

Since μ is the Gibbs state for the potential $\psi = 0$ on Λ_N, the measure of points that follow a fixed itinerary of length q (with respect to the Markov partition) is smaller than $C_2 \exp(-q \cdot h_{top}(\Lambda_N, f))$ for some $C_2 > 0$ uniform in q, see [Bow]. Using (17), we deduce that the number of different itineraries of length $q = \ell \cdot N$

starting from $(1-\beta)Q$ and contained in V_ℓ is larger than $C_3 \exp(q \cdot (h_{top}(\Lambda_N, f) - \varepsilon/N))$. Passing to the limit as q goes to $+\infty$ proves that the topological entropy of the maximal invariant set in $V_\ell \cup f(V_\ell) \cup \cdots \cup f^{N-1}(V_\ell)$ is larger than $h_{top}(\Lambda_N, f) - \varepsilon/N$, hence larger than $h_{top}(\Lambda, f) - \varepsilon$.

This completes the proof of Theorem 6.7. □

6.6. **Proof of Proposition 6.9.** The proof uses the chart metric. Consider any horseshoe Λ_N with N larger than some $N_0 \geq 1$, a chart φ whose connected components have diameter smaller than some $\rho_0 > 0$, a cube $Q \in \mathcal{Q} = \mathcal{Q}_{\varphi,\sigma,k}$ and any $x \in \Lambda_N \cap (1-2\beta)Q$.

We say that a cube $Q' \in \mathcal{Q}$ is *s-bad* (with respect to the image $f^N(Q)$) if its stable boundary $\partial^s_{\beta/2} Q'$ intersects $f^N(Q)$ (see Figure 8).

We will assume that the images of the unstable β-boundaries is larger than the size of the cubes:

$$\beta \exp(N_0 \widehat{\nu}) > 10. \tag{18}$$

Any cube $Q' \in \mathcal{Q}$ that intersects $f^N((1-\frac{3}{4}\beta)Q \cap W^u(x,\rho_0)) \cap \Lambda_N$ satisfies one of the following cases:

- either there exists a transition $Q \to Q'$,
- or Q' is s-bad.

(See Figure 9.) Indeed Q' has size 2^{-k} and by (18) cannot intersect $f^N(\partial^u_{\beta/2} Q)$, which is at distance larger than $\exp(N\widehat{\nu})\frac{\beta}{10}2^{-k}$ from $f^N((1-\frac{3}{4}\beta)Q)$.

In the following the main point is to bound the measure of s-bad cubes that intersect $f^N((1-\beta)Q \cap W^u(x,\rho_0)) \cap \Lambda_N$

6.6.1. *Measure of cube boundaries.* For the particular geometry of the cubes that we've defined, the reverse doubling inequality (13) can be improved.

Lemma 6.10. *For every $\delta > 0$, there is $\eta \in (0,1)$ such that for any $\sigma \in (0,1)$ the following property holds if ρ_0 is sufficiently small.*

For any $N \geq N_0$, any $x \in \Lambda_N$ and any cube $Q = Q_{r,\sigma}(y)$ such that $(1-3\sigma)Q$ and $W^u(x,\rho_0)$ intersect, the measure μ^u_x induced on $W^u(x,\rho_0)$ by the disintegration of the measure of maximal entropy of Λ_N satisfies:

$$\mu^u_x(Q \setminus (1-\eta)Q) \leq \delta \sum_{Q' \in \mathcal{N}^u(Q)} \mu^u_x((1-\eta)Q').$$

Proof. The proof is similar to that of Theorem 6.4. We first introduce for each point $\zeta \in \Lambda_N$ and each $k \geq 1$ the domains $\Delta_{\zeta,k}$ and $\Delta_{\zeta,k,\pm}$ as intersections of the k-th backward image of rectangles $f^{-k}(R_i)$ or $f^{-k}(R_{i,\pm})$ with the unstable plaque of ζ.

Sublemma 6.11. *For every $\eta_0 > 0$, there exists $\eta_1 \in (0,\eta_0)$ such that for any $\sigma \in (0,1)$ the following property holds if ρ_0 is sufficiently small.*

For any $N \geq N_0$, any cube $Q = Q_{r,\sigma}(y)$ and any $z \in (1-3\sigma)Q \cap \Lambda_N$, it holds that any point ζ in $\Lambda_N \cap (1+\eta_1)Q \setminus (1-\eta_1)Q$ belongs to some preimage $\Delta_{\zeta,k}$ such that:

1. *$\Delta_{\zeta,k} \subset (1+\eta_0)Q \setminus (1-\eta_0)Q$, and*
2. *$\Delta_{\zeta,k,-}$ or $\Delta_{\zeta,k,+}$ is contained in a cube $(1-\eta_1)Q'$ with $Q' \in \mathcal{N}^u(Q)$.*

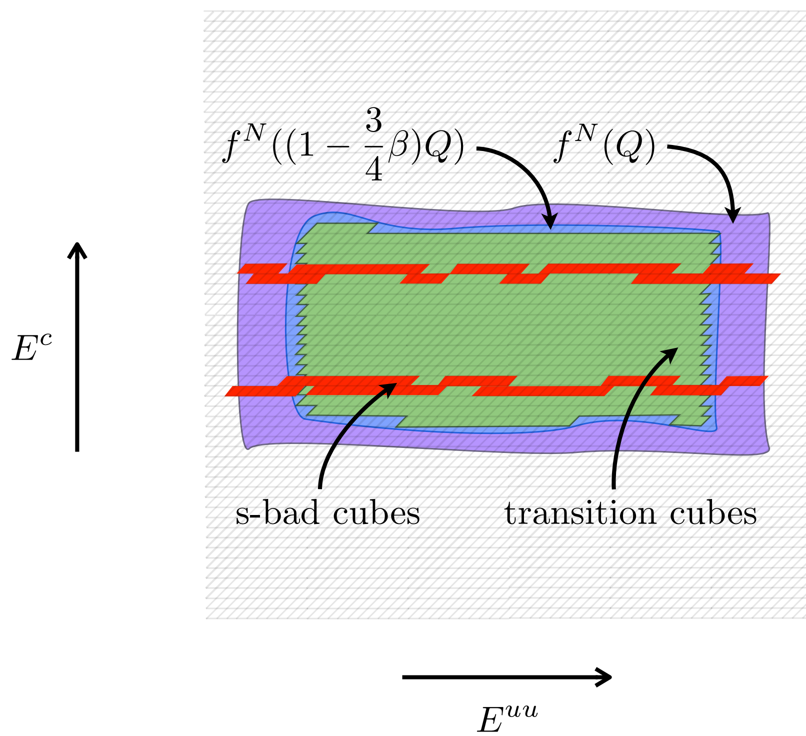

Figure 9. Transition cubes in $f^N(W^u(x,\rho_0))$.

Proof. Assuming that ρ_0 is small enough, the bundles E^s, E^c, E^{uu} (viewed in the charts) are C^0-close to constant bundles and the unstable plaques are C^1-close to affine spaces. The sets $\Delta_{\zeta,k}$ are inside the unstable plaque of z, which is stretched along a central curve γ_k and very thin in the strong unstable direction (see (14)).

Since $z \in (1-3\sigma)Q$, the plaque $W^u(z,\rho)$ intersects any $Q' \in \mathcal{N}^u(Q)$ along its unstable boundary. The intersection with each unstable face of Q' is transverse. It follows that the set $(1+\eta_0)Q \setminus \bigcup_{Q'\in\mathcal{N}^u(Q)}(1-\eta_1)Q'$ is a union of $2d^u$ thickened hypersurfaces $S_1,\dots,S_{2d^u}$ of the unstable plaque of z whose width along the central direction E^c belongs to $[\eta_1 r/2, 2\eta_1 r]$.

By (15), for any $\zeta \in \Lambda_N \cap (1+\eta_1)Q \setminus (1-\eta_1)Q$, one can consider a domain Δ_{ζ,k_0} associated to a central curve $f^{-k_0}(\gamma_{k_0})$ of length contained in $[2\eta_1 r C^2\frac{\widehat{L}}{L}, \kappa^{-1}\eta_1 r C^2\frac{\widehat{L}}{L}]$ (using the constants $C, L, \widehat{L}, \kappa > 0$ introduced in the proof of Theorem 6.4). The two domains $\Delta_{\zeta,k_0,+}$ and $\Delta_{\zeta,k_0,-}$ are associated to subintervals $I^-, I^+ \subset \gamma_{k_0}$ whose k_0-th preimages are separated by $C^{-2}\frac{L}{\widehat{L}}\mathrm{Length}(f^{-k_0}(\gamma_{k_0}))$, which is larger than $2\eta_1 r$. It follows that only one domain $\Delta_{\zeta,k_0,+}$ or $\Delta_{\zeta,k_0,-}$ can intersect each thickened hypersurface S_i. By the definition of split Markov partition, either $\Delta_{\zeta,k_0,+}$ or $\Delta_{\zeta,k_0,-}$ must contain both $\Delta_{\zeta,k_0+1,+}$ and $\Delta_{\zeta,k_0+1,-}$. We thus deduce that in the family $\Delta_{\zeta,k_0}, \dots \Delta_{\zeta,k_0+2d_u}$, there exists a domain $\Delta_{\zeta,k}$ such that either $\Delta_{\zeta,k,-}$ or $\Delta_{\zeta,k,+}$ is disjoint from the thickened hypersurfaces $S_1,\dots,S_{2d^u}$, and hence is contained in a cube $(1-\eta_1)Q'$ with $Q' \in \mathcal{N}^u(Q)$.

By (15), the larger domain $\Delta_{\zeta,k_0+2^{d_u}}$ is associated to a curve $\gamma_{k+2^{d_u}}$ whose length is smaller than $\left(\frac{1}{2\kappa}\right)^{2^{d_u}}\kappa^{-1}\eta_1 r C^2\frac{\widehat{L}}{L}$. It is thus contained in $(1+\eta_0)Q\setminus(1-\eta_0)Q$, provided η_1 is chosen so that

$$\eta_0 > \left(\frac{1}{2\kappa}\right)^{2d_u} C^2\frac{\widehat{L}}{L}\eta_1.$$

□

For $\eta_1<\eta_0$ as in the previous sublemma, any point ζ belongs to a maximal set Δ_ζ satisfying conditions 1 and 2. In particular, the domains Δ_ζ are disjoint or equal, they cover $\Lambda_N\cap(1+\eta_1)Q\setminus(1-\eta_1)Q$, and they satisfy

$$\mu_x^u(\Delta_\zeta)\le\frac{1}{2}\sum_{Q'\in\mathcal{N}^u(Q)}\mu_x^u(\Delta_\zeta\cap(1-\eta_1)Q').$$

We obtain that

$$\mu_x^u((1+\eta_1)Q\setminus(1-\eta_1)Q)\le\frac{1}{2}\sum_{Q'\in\mathcal{N}^u(Q)}\mu_x^u((1-\eta_1)Q'\cap(1+\eta_0)Q\setminus(1-\eta_0)Q).$$

Applying the sublemma inductively, we construct a sequence $0<\eta_k<\eta_{k-1}<\cdots<\eta_0<1$ such that $2^{-k}<\delta$ and

$$\mu_x^u((1+\eta_\ell)Q\setminus(1-\eta_\ell)Q)\le 2^{-\ell}\sum_{Q'\in\mathcal{N}^u(Q)}\mu_x^u((1-\eta_\ell)Q'\cap(1+\eta_{\ell-1})Q\setminus(1-\eta_{\ell-1})Q),$$

for $\ell=0,\dots,k$. Thus for $\eta=\eta_k$, we have:

$$\mu_x^u(Q\setminus(1-\eta)Q)\le\delta\sum_{Q'\in\mathcal{N}^u(Q)}\mu_x^u((1-\eta_1)Q'),$$

which implies the conclusion of Lemma 6.10. □

6.6.2. *Localization of s-bad cubes.* In order to control the image $f^N(Q)$ of a cube, we require that it be smaller than ρ_1 (and contained in the domain of the chart φ):

$$\exp(N_0\widehat{\lambda})\,2^{-k_0}<\rho_1, \tag{19}$$

and that its diameter (in the chart) along the coordinate space $\{0\}^{d_u}\times\mathbb{R}^{d^s}$ be less than $\frac{\beta}{2}2^{-k}$:

$$\exp(-N_0\nu)<\frac{\beta}{2}. \tag{20}$$

We define a *strong unstable strip* of an unstable plaque $W^u(z,\rho_0)$, $z\in\Lambda$, as the region bounded by two strong unstable leaves $W^{uu}(z_1,2\rho_0)$, $W^{uu}(z_2,2\rho_0)$ with $z_1,z_2\in W^u(z,\rho_0)$: this is the set of points $\zeta\in W^u(z,\rho_0)$ that belong to a central curve in $B(z,2\rho_0)$ joining $W^{uu}(z_1,2\rho_0)$ to $W^{uu}(z_2,2\rho_0)$.

Fix a continuous central cone field, that is, at each point x close to Λ a cone in T_xM that is a small neighborhood of $E^c(x)$. The distance between two different W^{uu}-leaves in $W^u(y,\rho_0)$ is the infimum of the length of curves joining the two leaves and tangent to the central cone field. This allows us to define the width of a strong unstable strip and the distance between two strong unstable strips. Note that the strong unstable manifolds W^{uu} form a $C^{1+\alpha}$ subfoliation of any unstable

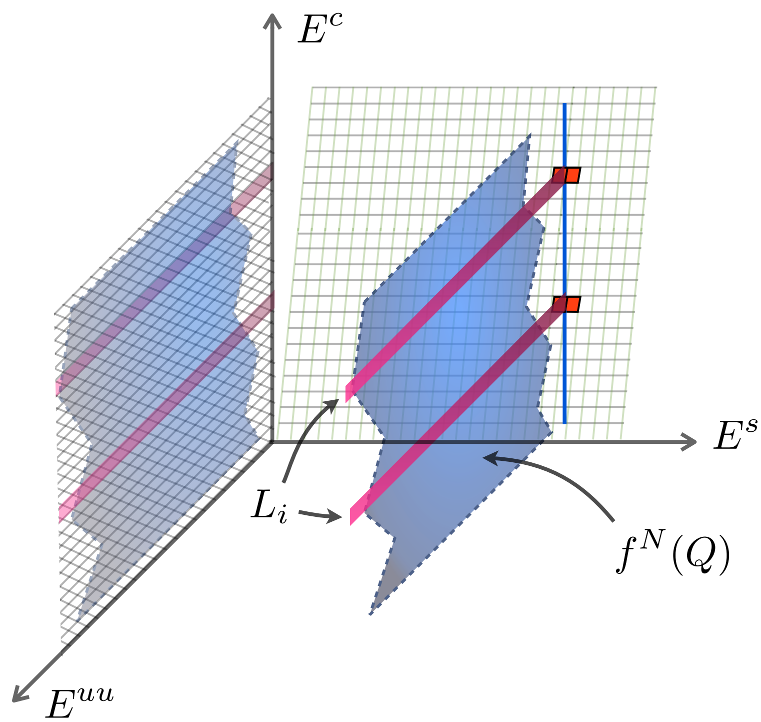

FIGURE 10. Covering s-bad cubes by strong unstable strips.

manifold $W^u(y)$, so if ρ_0 is small and the central cones are thin, the length of the central curves differ by a multiplicative constant, which we could take to be as close to 1 as we want.

The next statement asserts that s-bad cubes are contained in a union of strips $L_1, \dots, L_q$ (see Figure 5) that are well separated by a distance $r > 0$ and all have the same width ηr where $\eta \in (0, 1/2)$ is chosen small, but large enough relative to β so that that the following condition is satisfied:

$$\beta < d_s^{-2d_s} \left(\frac{\eta}{100}\right)^{(d_s+1)}. \tag{21}$$

Lemma 6.12. *For any $\beta, \sigma, \eta \in (0, 1)$, $N_0 \geq 1$ satisfying* (20) *and* (21)*, and such that $\sigma < \beta/2$, the following property holds if ρ_0 is sufficiently small.*

For any $N \geq N_0$, choose k_0 satisfying (19)*. For any $k \geq k_0$, $Q \in \mathcal{Q}_{\varphi,k,\sigma}$ and any $x \in Q \cap \Lambda_N$, there exist strong unstable strips $L_1, \dots, L_q$ in $W^u(f^N(z), \rho_0)$ and $r \in [d_s^{-2d_s} (\frac{\eta}{100})^{d_s} \frac{2^{-k}}{\sigma}, \frac{2^{-k}}{\sigma}]$ such that:*

- *each strip has width less than $\frac{\eta r}{4}$;*
- *the distance between two strips is greater than $4r$;*
- *the intersection of each s-bad cube $Q' \in \mathcal{Q}_{\varphi,k,\sigma}$ with $f^N(W^u(x, \rho_0))$ is contained in the union of the strips $\bigcup_j L_j$.*

Proof. By (19), the image $f^N(Q)$ has diameter smaller than ρ_1, and the projection of $f^N(Q)$ on the space $\{0\}^{d_u} \times \mathbb{R}^{d_s}$ (inside the chart φ) has diameter less than

$\frac{\beta}{2}2^{-k}$. In particular, for any point $y \in f^N(Q)$, there exist $z \in W^u(f^N(x), \rho_0)$ such that $\varphi(z) - \varphi(y)$ belongs to $\{0\}^{d_u} \times \mathbb{R}^{d_s}$ and has norm less than $\frac{\beta}{2}2^{-k}$. It follows that the s-bad cubes Q' that intersect the plaque $f^N(W^u(x, \rho_0))$ are a subset of those $Q' \in \mathcal{Q}$ satisfying:

$$\partial^s_\beta Q' \cap W^u(f^N(x), \rho_0) \neq \emptyset.$$

In the charts, the plaque $W^u(f^N(x), \rho_0)$ is close to a plane H parallel to $\mathbb{R}^{d_u} \times \{0\}^{d_s}$. The intersection $H \cap \bigcup_{Q' \in \mathcal{Q}} \partial^s Q'$ is contained in a finite union Z of hyperplanes parallel to $\mathbb{R}^{d_u - 1} \times \{0\}^{d_s+1}$ in H. By projecting these hyperplanes on the e_{d_u}-axis, one obtains a set X of points that is contained in the translates of (at most) d_s points under $(2^k\sigma)^{-1} \cdot \mathbb{Z}$. One can thus project X to a subset $\overline{X}$ of $\mathbb{R}/(2^k\sigma)^{-1} \cdot \mathbb{Z}$ of cardinality d_s and then apply the following elementary lemma with $a = \frac{\eta}{50}$.

Sublemma 6.13. *Let $X \subset \mathbb{R}/\mathbb{Z}$ be a finite subset of the circle and $d = \#X$. Then for every $a \in (0, 1/2)$, there exists $\kappa \in [d^{-2d}(a/2)^d, a/2]$ and a collection $I_1, \dots, I_n \subset \mathbb{R}/\mathbb{Z}$ of open intervals with the following properties:*

- $X \subset I_1 \cup \dots \cup I_n$*;*
- $\mathrm{Length}(I_j) = \kappa$*, for* $j = 1, \dots, n$*;*
- *the length of the connected components of* $(\mathbb{R}/\mathbb{Z}) \setminus (I_1 \cup \dots \cup I_n)$ *is larger than* κ/a*.*

Proof. The proof is by induction on d. The cases $d = 1$ and $d = 2$ are easy. For $d > 2$, let us set $\ell = d^{-2d}(a/2)^d$. If the minimum distance between distinct points in X is at least $(1 + a^{-1})\ell$, then we put an interval of diameter $\kappa := \ell$ centered at each point of X and the conclusion holds.

Otherwise we collapse all the connected components of $\mathbb{R}/\mathbb{Z}$ of length less or equal to $(1 + a^{-1})\ell$. (There are at most $d - 1$ many such components.) We get a circle of length $L < 1$ with at most $d - 1$ points. Rescaling the quotient circle to unit length and applying the inductive hypothesis with $a' = \frac{d-2}{d-1}a$, we obtain a collection of intervals $I'_1, \dots, I'_n$ of length κ' and separated by intervals of length larger than κ'/a'.

Pulling back the intervals I'_j to the initial circle, we obtain intervals $\widetilde{I}_j$ of length $\kappa' L + \ell_j$ where ℓ_j is the sum of the lengths of the intervals contained in $\widetilde{I}_j$ that have been collapsed. One can enlarge the intervals $\widetilde{I}_j$ and get intervals $I_1, \dots, I_n$ of length $\kappa := \kappa' L + (d - 1)(1 + a^{-1})\ell$ and separated by distances larger than $\frac{\kappa'}{a'}L - (d - 1)(1 + a^{-1})\ell$.

By definition, $\kappa \geq \ell$. Using $L \leq 1$, $\kappa' \leq a'/2$, $(1+a^{-1})a/2 < 1$ and the definitions of κ, ℓ and a', one gets easily $\kappa \leq a/2$.

In order to check that the intervals I_j are separated by κ/a, we estimate

$$\left(\frac{\kappa'}{a'}L - (d-1)(1+a^{-1})\ell\right) - \kappa/a = \frac{2(d-1)L\ell}{a^2}\left(\frac{\frac{a}{2}\kappa'}{(d-1)(d-2)\ell} - \frac{(1+a)^2}{2L}\right). \tag{22}$$

The induction assumption gives $\kappa' \geq (d-1)^{-2(d-1)} \left(\frac{a'}{2}\right)^{d-1}$ and together with the definitions of a' and ℓ we get:

$$\frac{\frac{a}{2}\kappa'}{(d-1)(d-2)\ell} \geq \frac{d^{2d}(d-2)^{d-2}}{(d-1)^{3d-2}} > \exp(1).$$

On the other hand, $\frac{(1+a)^2}{2L}$ is smaller than $3/2$, since $a < 1/2$. It follows that (22) is positive, which concludes the proof. $\square$

From the Sublemma 6.13 we obtain $\kappa \in [d_s^{-2d_s}(a/2)^{d_s}, a/2]$ and some subintervals $I_1, \dots, I_n$ of the circle. Pulling back the intervals I_i to the e_{d_u}-axis, one can extract a finite collection of intervals $J_1, \dots, J_q$ of length $\frac{\kappa}{\sigma}2^{-k}$, separated by $\frac{\kappa}{a\sigma}2^{-k}$ and whose union contains X.

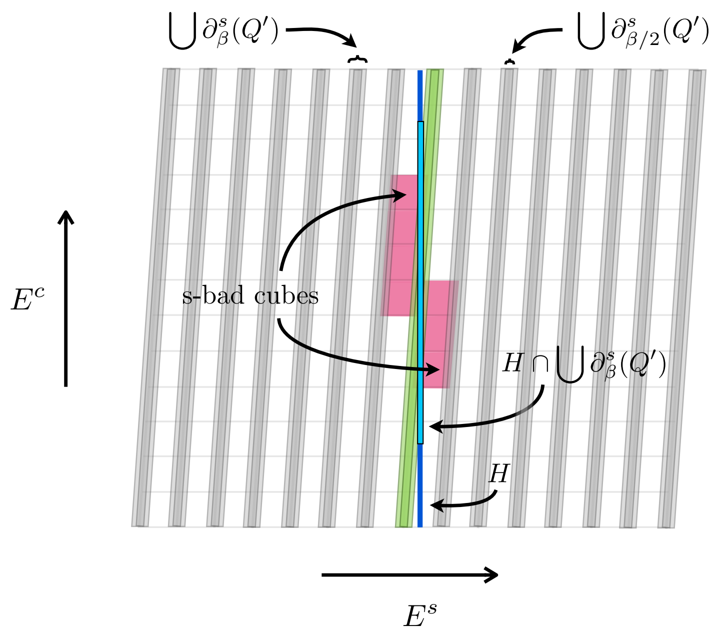

FIGURE 11. $H \cap \bigcup_{Q' \in \mathcal{Q}} \partial^s_\beta Q'$ contains the intersection of H with the s-bad cubes for Q.

Consider the intersection $H \cap \bigcup_{Q' \in \mathcal{Q}} \partial^s_\beta Q'$ (see Figure 11). On the one hand, since $\sigma < \beta/2$, it contains the intersection of H with the s-bad cubes for Q. On the other hand, $H \cap \bigcup_{Q' \in \mathcal{Q}} \partial^s_\beta Q'$ is contained in the $\frac{2\beta}{\sigma}2^{-k}$-neighborhood of Z, hence in q strips of H of width smaller than $\frac{3}{2}\frac{\kappa}{\sigma}2^{-k}$ (since by our choice of β we have $\beta < \kappa/5$) and separated by $\frac{9}{10}\frac{\kappa}{a\sigma}2^{-k}$. We obtain the strips $L_1, \dots, L_q$ by projection of the strips of H to the plaque $W^u(f^N(x), \rho_0)$ and set $r = \frac{\kappa}{5a\sigma}2^{-k}$: the strips are separated by a distance larger than $4r$ and have width smaller than $10ar$, which is smaller than $\frac{\eta r}{4}$ by our choice of a. We deduce the estimates on r from the bounds on κ. $\square$

We then cover the strips obtained from Lemma 6.12 by a collection of unstable η-boundaries of cubes C of size r (see Figure 12). We require the cubes C to be smaller than the unstable β-boundary of the image cube $f^N(Q)$:

$$\frac{\eta}{\sigma} < \tfrac{1}{100}\beta \exp(N_0\widehat{\nu}). \tag{23}$$

Corollary 6.14. *In the setting of Lemma 6.12, assume that* (23) *holds. Then there exists a collection of cubes* $C_1 = Q_{r,\sigma}(y_1), \ldots, C_n = Q_{r,\sigma}(y_n)$ *such that:*

(1) *each cube* C_i *is disjoint from* $f^N(\partial^u_{\beta/2}Q)$*;*
(2) *each set* $C_i \cap W^u(f^N(x), \rho_0)$ *is non-empty and contained in* $f^N(Q)$*;*
(3) $(L_1 \cup \cdots \cup L_m) \cap f^N((1-\frac{3}{4}\beta)Q) \subset \bigcup_i C_i$*;*
(4) *if* C_i *intersects* $(L_1 \cup \cdots \cup L_m) \cap f^N((1-\frac{3}{4}\beta)Q)$*, then* $\mathcal{N}^u(C_i) \subset \{C_1, \ldots, C_n\}$*;*
(5) *the interiors of the cubes* C_i *are pairwise disjoint;*
(6) *the inner cubes* $(1-\eta/2)C_i$ *and the strips* L_j *do not intersect.*

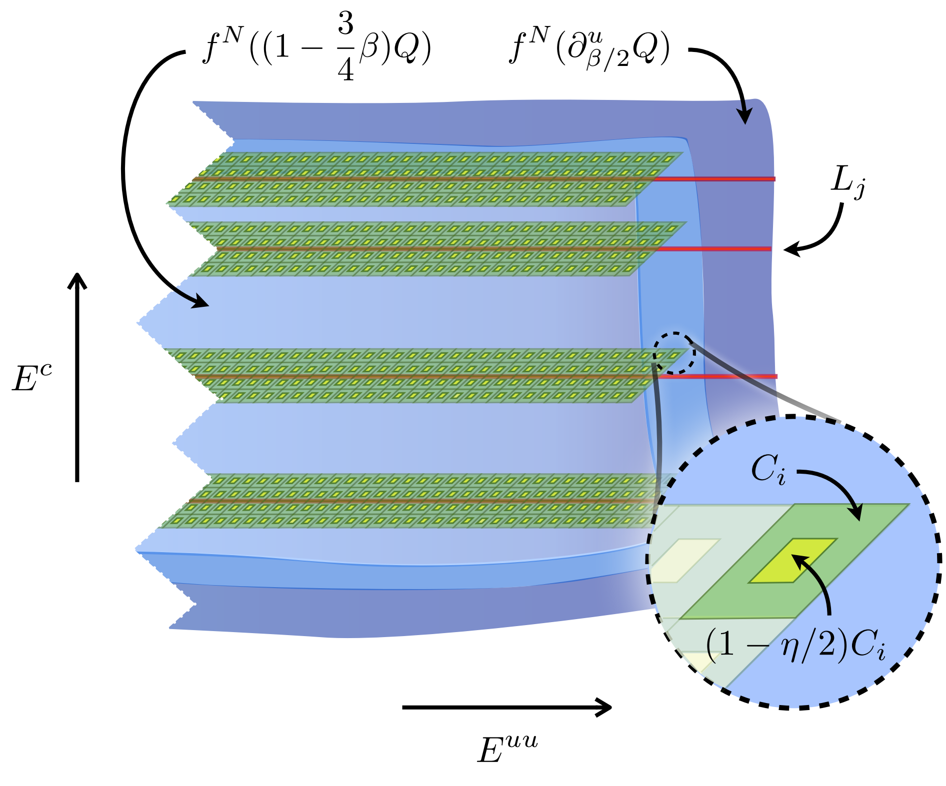

FIGURE 12. Covering strips in $f^N(Q)$ with (larger) cubes C_i in order to bound their measure.

Proof. For each strip L_j, we choose a point y_0 in L_j and (in the chart φ) we consider the cubes of the form

$$C = Q_{r,\sigma}(y), \;\; \text{with } y \in \varphi^{-1}(\varphi(y_0) + rL_\sigma(\mathbb{Z}^d + \tfrac{1}{2}e_{d_u}))$$

that either intersect the strip $L_j \cap f^N((1-\frac{3}{4}\beta)Q)$ or that have an unstable neighbor cube that intersects this strip. By construction, the union of these cubes C contains $L_j \cap f^N((1-\frac{3}{4}\beta)Q)$.

By the inequality (23), the size $r \le \frac{\eta}{2\sigma}2^{-k}$ of the cubes C is much smaller than the distance between $f^N((1-\frac{3}{4}\beta)Q)$ and $f^N(\partial^u_{\beta/2}Q)$ in the plaque $W^u(f^N(z),\rho_0)$. This implies item (1). Moreover the intersection of each cube C with $W^u(f^N(z),\rho_0)$ is contained in $f^N(Q)$.

Recall that L_j is C^1-close to a plane spanned by $e_1,\dots,e_{(d_u-1)}$ and has width smaller than $\eta r/4$. Note also that y_0 belongs to the center of an unstable face of some adjacent cubes C_0, C_0'. Any cube C that intersects L_j is the image of C_0 or C_0' by a translation by a bounded vector in $\mathbb{Z}e_1+\dots+\mathbb{Z}e_{(d_u-1)}$: it also intersects L_j near the center of an unstable face. The other cubes C are unstable neighbor cubes of those who intersect L_j. In particular, item (2) holds. Since the width of L_j is smaller than $\eta r/4$, we deduce that the inner cubes $(1-\eta/2)C$ do not intersect the strip L_j.

The collection $\{C_1,\dots,C_n\}$ is the union of the collection of cubes C associated to each strip L_j. Since the distance between the strips is larger than $4r$, the cubes associated to L_j do not intersect the cubes associated to $L_{j'}$ if $j\neq j'$. The other items follow. $\square$

6.6.3. *End of the proof of Proposition 6.9.* Given $\varepsilon>0$, we make the following choices:

- $\delta \in \big(0, 3^{-(d_u+1)}(e^{\varepsilon/2}-1)\big)$ controls the μ^u-measure of poorly crossed cubes.
- The boundary size $\eta=\eta(\delta)$ is chosen according to Lemma 6.10.
- The boundary size β is chosen small in order to satisfy (21), and the shear is fixed to be $\sigma=\beta/10$.
- The iterate N is larger than a bound N_0 satisfying the properties of Section 6.4.2, and the inequalities (18), (20) and (23).
- The scales ρ_0 and $\rho_1 \ll \rho_0$ are chosen (independently from N_0 and N) so that the variations of the splitting $E^{uu}\oplus E^c\oplus E^s$ are small (see Section 6.4.3, Lemmas 6.10 and 6.12).
- A lower bound k_0 on the scales k, chosen so that f^N is nearly linear at scale 2^{-k_0} and such that (19) holds.

Lemma 6.15. *Under these assumptions, the union of the s-bad cubes in $\mathcal{Q}$ that intersect $f^N((1-\frac{3}{4}\beta)Q)$ has $\mu^u_{f^N(x)}$ measure smaller than*

$$3^{(d_u+1)}\delta \sum_{Q\to Q'} \mu^u_{f^N(x)}((1-\eta)Q').$$

Proof. By Lemma 6.12, and (18) (the size of the cube in $\mathcal{Q}$ is much smaller than the unstable β-boundary of $f^N(Q)$), we aim to bound the $\mu^u_{f^N(x)}$-measure of

$$\Delta := (L_1\cup\dots\cup L_m)\cap f^N((1-\tfrac{3}{4}\beta)Q).$$

By Corollary 6.14, items (3) and (6), it is bounded by:

$$\mu^u_{f^N(x)}(\Delta) \le \sum_{C_i\cap\Delta\neq\emptyset} \mu^u_{f^N(x)}(C_i\setminus(1-\eta)C_i).$$

By Lemma 6.10, the measure of the η-boundary of each C_i is smaller than

$$\mu^u_{f^N(x)}(C_i \setminus (1-\eta)C_i) \leq \delta \sum_{C \in \mathcal{N}^u(C_i)} \mu^u_{f^N(x)}((1-\eta)C).$$

If $C_i \cap \Delta \neq \emptyset$, then the cubes $C \in \mathcal{N}^u(C_i)$ still belong to the family $\{C_1, \dots, C_n\}$ (by Corollary 6.14 item 4); moreover each cube C belongs to no more than 3^{d_u} sets $\mathcal{N}^u(C_i)$. This gives

$$\mu^u_{f^N(x)}(\Delta) \leq 3^{d_u}\ \delta \sum_{i=1}^{n} \mu^u_{f^N(x)}((1-\eta)C_i).$$

Consider any cube $Q' \in \mathcal{Q}$ that intersects $(1-\eta)C_i \cap W^u(f^N(x), \rho_0)$ and take $Q'' \in \mathcal{N}^u(Q')$. Then we claim that $Q'' \subset (1-\frac{\eta}{2})C_i$ and $Q \to Q''$. Indeed:

- $Q'' \subset (1-\frac{\eta}{2})C_i$, since from $\sigma = \frac{\beta}{10}$ and (21) the size of the cubes in $\mathcal{Q}$ is smaller than the η-boundary size of the cubes C_i.
- Q'' is thus disjoint from L_i and is not an s-bad cube by Corollary 6.14 item 6.
- Q'' and $W^u(f^N(x), \rho_0)$ intersect: since Q' intersects $W^u(f^N(x), \rho_0)$, is not an s-bad cube, and since $\sigma = \beta/10$, the inner cube $(1-3\sigma)Q'$ and $W^u(f^N(x), \rho_0)$ intersect; this implies that the unstable neighbor cube Q'' also intersect $W^u(f^N(x), \rho_0)$.
- Q'' intersects $f^N(Q)$, since $Q'' \cap W^u(f^N(x), \rho_0)$ is non-empty and contained in $C_i \cap W^u(f^N(x), \rho_0)$, which is contained in $f^N(Q)$ by Corollary 6.14 item 2.
- $Q'' \subset C_i$ does not meet $f^N(\partial^u_{\beta/2} Q)$ by corollary 6.14 item 1.

Applying Lemma 6.10 to the cubes $Q' \in \mathcal{Q}$ that intersect $(1-\eta)C_i \cap W^u(f^N(x), \rho_0)$ it follows that

$$\mu^u_{f^N(x)}((1-\eta)C_i) \leq (1 + 3^{d_u}\ \delta) \sum_{Q'' \in \mathcal{Q}(C_i)} \mu^u_{f^N(x)}((1-\eta)Q''),$$

where $\mathcal{Q}(C_i)$ is the family of cubes $Q'' \in \mathcal{Q}$ such that $Q \to Q''$ and whose intersection with $W^u(f^N(x), \rho_0)$ is non-empty and contained in $(1-\frac{\eta}{2})C_i$.

By Corollary 6.14 item 5, the sets $(1-\frac{\eta}{2})Q_i$ are disjoint. This gives

$$\mu^u_{f^N(x)}(\Delta) \leq 3^{d_u}\ \delta(1 + 3^{d_u}\ \delta) \sum_{Q \to Q'} \mu^u_{f^N(x)}((1-\eta)Q'),$$

which implies the result since δ is smaller than 3^{-d_u}. □

We now bound the measure:

$$\mu^u_{f^N(x)}(f^N((1-\beta)Q)) \leq \sum_{Q' \in \mathcal{Q}(x)} \mu^u_{f^N(x)}(Q')$$

where $\mathcal{Q}(x)$ is the collection of $Q' \in \mathcal{Q}$ that intersect $f^N((1-\beta)Q) \cap W^u(f^N(x), \rho_0)$.

The last sum may be decomposed in three parts:

(1) We first consider the cubes Q' that are s-bad: Lemma 6.15 implies
$$\sum_{Q'\in\mathcal{Q}(x)\text{ s-bad}} \mu^u_{f^N(x)}(Q') \leq 3^{(d_u+1)}\delta \sum_{Q\to Q'} \mu^u_{f^N(x)}((1-2\beta)Q').$$

(2) We then consider the inner part $(1-2\beta)Q'$ of the cubes that are not s-bad. As explained at the beginning of Section 6.6 we necessarily have $Q \to Q'$.
$$\sum_{Q'\in\mathcal{Q}(x)\text{ not s-bad}} \mu^u_{f^N(x)}((1-2\beta)Q') \leq \sum_{Q\to Q'} \mu^u_{f^N(x)}((1-2\beta)Q').$$

(3) We finally consider the boundary part $Q'\setminus(1-2\beta)Q'$ of the cubes that are not s-bad. Since $3\sigma < 2\beta$, Lemma 6.10 applies and gives
$$\sum_{\substack{Q'\in\mathcal{Q}(x)\\ \text{not s-bad}}} \mu^u_{f^N(x)}(Q'\setminus(1-2\beta)Q') \leq \delta \sum_{\substack{Q'\in\mathcal{Q}(x)\\ \text{not s-bad}}} \sum_{Q''\in\mathcal{N}^u(Q')} \mu^u_{f^N(x)}((1-2\beta)Q'').$$

Note that a cube Q'' appears at most 3^{d_u} times in the last sum. Moreover since it is an unstable neighbor of a cube Q' that is not s-bad and intersects $f^N((1-\beta)Q)\cap W^u(f^N(x),\rho_0)$, the cube Q'' must also intersect $f^N((1-\frac{3}{4}\beta)Q)\cap W^u(f^N(x)\rho_0)$. One may thus again distinguish those that are s-bad, where we use Lemma 6.15, from those that are not s-bad and satisfy $Q\to Q''$. This gives:
$$\sum_{\substack{Q'\in\mathcal{Q}(x)\\ \text{not s-bad}}} \mu^u_{f^N(x)}(Q'\setminus(1-2\beta)Q') \;\leq\; 3^{d_u}\delta\;(1+3^{(d_u+1)}\delta)\sum_{Q\to Q'} \mu^u_{f^N(x)}((1-2\beta)Q').$$

Combining these estimates gives:
$$\mu^u_{f^N(x)}(f^N((1-\beta)Q)) \leq (1+3^{(d_u+1)}\delta)^2 \sum_{Q\to Q'} \mu^u_{f^N(x)}((1-2\beta)Q').$$

By our choice of δ we have $(1+3^{(d_u+1)}\delta)^2 < e^{\varepsilon}$. One gets the estimate of Proposition 6.9 since f^N has a constant Jacobian along the unstable leaves for the measure μ^u.

7. Birth of blenders inside large horseshoes

In this section we prove Theorem E.

7.1. **The recurrent compact criterion for horseshoes.**

Let Λ be a horseshoe for a C^1 diffeomorphism f whose unstable bundle has a dominated splitting $E^u = E^{uu}\oplus E^c$. Let d_{uu}, d_u, d_s be the strong unstable, unstable and stable dimensions and let $d_{cs} = (d_u - d_{uu}) + d_s$ the dimension of the bundle $E^c\oplus E^s$.

The local strong unstable lamination $\mathcal{W}^{uu}_{loc}$. For any g that is C^1-close to f and any x in the continuation Λ_g, there exist local unstable manifolds $W^u_{loc}(g,x)$, which depend continuously on (x,g) in the C^1 topology and which satisfy
$$g(W^u_{loc}(g,x)) \supset \overline{W^u_{loc}(g,g(x))}.$$
Each local unstable manifold supports a strong unstable foliation tangent to E^{uu} and for each $z\in W^u_{loc}(g,x)$ we denote by $W^{uu}_{loc}(g,z)$ the (connected) leaf containing z. The collection of all local strong unstable leaves defines the local strong unstable lamination $\mathcal{W}^{uu}_{loc}(g)$ associated to Λ_g. Since Λ is totally disconnected we may assume

furthermore that for any $x, y \in \Lambda_g$ the plaques $W^u_{loc}(g,x), W^u_{loc}(g,y)$ are either disjoint or equal.

The transversal $\mathcal{D}$. We then consider a submanifold $\mathcal{D} \subset M$ such that for each $x \in \Lambda$, the intersection between the closures of $\mathcal{D}$ and $W^u_{loc}(x)$ is contained in $\mathcal{D} \cap W^u_{loc}(x)$, is transverse to the strong unstable foliation and contains at most one point of each strong unstable leaf. In particular these properties still hold for $\mathcal{D}$ and the local strong unstable lamination $\mathcal{W}^{uu}_{loc}(g)$ if g is C^1-close to f.

The following definition comes from [MS] and generalizes a property introduced in [MY] for producing robust intersections of regular Cantor sets.

Definition 7.1. A compact set $K \subset \mathcal{D}$ is *recurrent* (with respect to $\mathcal{W}^{uu}_{loc}$) if for any points $x \in \Lambda$ and $z \in W^u_{loc}(x) \cap K$, there exists $n \geq 1$, $x' \in \Lambda$ and some point $z' \in W^u_{loc}(x')$ contained in the interior of K (in the topology of $\mathcal{D}$), such that $f^{-n}(x') \in W^u_{loc}(x)$ and $f^{-n}(\overline{W^{uu}_{loc}(z')}) \subset W^{uu}_{loc}(z)$.

This property is robust and implies that Λ is a d_{cs}-stable blender, as stated in the two next propositions (which also come from [MS]):

Proposition 7.2. *The compact set K is still recurrent with respect to the strong unstable leaves $\mathcal{W}^{uu}_{loc}(g)$ for the diffeomorphisms g that are C^1-close to f.*

Proof. Consider the (closed) set $I_g = \{(x,z) \in \Lambda_g \times K, z \in W^u_{loc}(g,x)\}$: it is contained in a small neighborhood of I_f if g is C^1-close to f. For any $(x_0, z_0) \in \Lambda_f$, consider $(x'_0, z'_0) \in \Lambda \times \mathcal{D}$ and $n \geq 1$ such that $f^{-n}(x'_0) \in W^u_{loc}(x_0)$ and $f^{-n}(W^{uu}_{loc}(z'_0)) \subset W^{uu}_{loc}(z_0)$. Then for any g that is C^1-close to f and any $(x,z) \in I_g$ that is close to (x_0, z_0), one can consider $x' \in \Lambda_g$ that is close to x'_0 such that $g^{-n}(x') \in W^u_{loc}(g,x)$. Since z'_0 belongs to $\mathcal{D}$, by continuity there exists $z' \in \mathcal{D} \cap W^u_{loc}(g,x')$ such that $g^{-n}(W^{uu}_{loc}(g,z')) \subset W^{uu}_{loc}(g,z)$. By compactness, there exists a small C^1 neighborhood $\mathcal{U}$ of f such that this property holds for any $(x_0, z_0) \in I_f$ for any $g \in \mathcal{U}$. □

Proposition 7.3 (Recurrent compact criterion)**.** *Consider a horseshoe Λ with a strong unstable bundle of codimension d_{cs}. If Λ admits a recurrent compact set K (with respect to $\mathcal{W}^{uu}_{loc}$) which intersects at least one plaque $W^u_{loc}(x)$ then it is a d_{cs}-stable blender.*

Proof. We start with preliminary considerations:

(1) By compactness, the integer $n \geq 1$ in Definition 7.1 may be chosen to be bounded by some n_0.
(2) Changing the metric, we may assume that $\|Df_{|E^s}(x)\| < 1$ at each point x of Λ. Iterating backwards, we may also assume that the local unstable manifolds are arbitrarily small. Since Λ is totally disconnected, there exists a smooth foliation in a neighborhood whose leaves have tangent spaces close to the bundle E^s. This defines local projections onto local unstable manifolds.
(3) The dominated splitting $E^{uu} \oplus (E^c \oplus E^s)$ gives the existence of an invariant cone field $\mathcal{C}^{uu}$: for each x in a neighborhood of Λ, the cone $\mathcal{C}^{uu}_x \setminus \{x\}$ is open in T_xM, and $Df_x(\overline{\mathcal{C}^{uu}(x)}) \subset \mathcal{C}^{uu}(f(x))$. Moreover, $\mathcal{C}^{uu}_x$ contains E^{uu}_x and is transverse to $E^c \oplus E^s$. Replacing the cone field by a forward iterate, we obtain an arbitrarily thin cone field around the bundle E^{uu} and defined

on a neighborhood of the lamination $\mathcal{W}^{uu}$. For any g that is C^1-close to f, the cone field is still invariant.

Let Γ be a *submanifold C^1-close to a local strong unstable manifold* $W^{uu}_{loc}(z)$ where $z \in K \cap W^u_{loc}(x)$ and $x \in \Lambda$. More precisely, this means that there exists a small constant $\varepsilon > 0$ such that Γ is tangent to $\mathcal{C}^{uu}$, its distance to $W^u_{loc}(x)$ (measured along the leaves of $\mathcal{F}$) is smaller than ε and its projection to $W^u_{loc}(x)$ contains z.

Claim. *Let g be a diffeomorphism C^1-close to f. Assume that Γ is close to the local strong unstable manifold $W^{uu}_{loc}(z)$ where $z \in K \cap W^u_{loc}(x)$ and $x \in \Lambda$. Consider $x' \in \Lambda$, $n \in \{1, \dots, n_0\}$, and $z' \in W^u_{loc}(x') \cap \operatorname{interior}(K)$ such that $f^{-n}(\overline{W^{uu}_{loc}(z')}) \subset W^{uu}_{loc}(z)$. Then $g^n(\Gamma)$ contains an open submanifold Γ' C^1-close to $\overline{W^{uu}_{loc}(z'')}$, for some $z'' \in W^u_{loc}(x') \cap \operatorname{interior}(K)$.*

Proof. Since $f^{-n}(\overline{W^{uu}_{loc}(z')}) \subset W^{uu}_{loc}(z)$ and n is bounded, there exists $\Gamma' \subset g^n(\Gamma)$ whose closure is close to $\overline{W^{uu}_{loc}(z')}$in the Hausdorff topology. From (3), it is tangent to the cone field $\mathcal{C}^{uu}$ and from (2), its distance to $W^u_{loc}(x')$ is smaller than ε. The projection of Γ' to $W^u_{loc}(x')$ intersects $\mathcal{D}$ at a point z'' close to z'. Since z' belongs to the interior of K, the point z'' belongs to interior(K) also. Consequently Γ' is C^1-close to $\overline{W^{uu}_{loc}(z'')}$ and the claim is proved. $\square$

Now consider g and Γ as in the claim. Applying the claim inductively, we get a decreasing sequence of submanifolds Γ_k and an increasing sequence of integers $n_k \to +\infty$ such that each $g^{n_k}(\Gamma_k)$ is close to a local strong unstable manifold. This proves that the forward orbit of the point $y = \cap_k \Gamma_k$ remains in a small neighborhood of Λ_g. Consequently, y belongs to the intersection of Γ and a local stable manifold of a point of Λ_g. This proves that Λ is a d_{cs}-blender. $\square$

7.2. **The recurrent compact criterion for affine horseshoes.**

Let $B^u = [0,1]^{d_u}$, $B^s = [0,1]^{d_s}$. We now assume that Λ satisfies the following additional property.

Definition 7.4. We say that Λ is a *standard affine horseshoe* if there exist:

- a chart $\varphi\colon U \to \mathbb{R}^{d_u+d_s}$ with $[0,1]^{d_u+d_s} \subset \varphi(U)$,
- a linear map $A \in \mathrm{GL}(d_u + d_s, \mathbb{R})$ which is a product $A^u \times A^s$ where $(A^u)^{-1}$ and A^s are contractions of $\mathbb{R}^{d_u}$ and $\mathbb{R}^{d_s}$,
- pairwise disjoint translated copies $B^s_1, \dots, B^s_\ell$ of $A^s(B^s)$ in interior(B^s),
- pairwise disjoint translated copies $B^u_1, \dots, B^u_\ell$ of $(A^u)^{-1}(B^u)$ in interior(B^u),

with the following properties:

- f sends $\varphi^{-1}(B^u_j \times B^s)$ to $\varphi^{-1}(B^u \times B^s_j)$ for each $1 \le j \le \ell$,
- $\varphi \circ f \circ \varphi^{-1}$ agrees on $B^u_j \times B^s$ with the map $z \mapsto A(z) + v_j$, for some $v_j \in \mathbb{R}^d$, and
- Λ is the maximal invariant set in $\bigcup \varphi^{-1}(B^u_j \times B^s)$.

Note that, denoting $R := \varphi^{-1}([0,1]^{d_u+d_s})$, the cubes $R_j := \varphi^{-1}(B^u_j \times B^s)$ are the connected components of $R \cap f^{-1}(R)$ that intersect Λ. See Figure 13.

We will assume furthermore that Λ is partially hyperbolic: A preserves the dominated splitting $\mathbb{R}^{d_s+d_u} = \mathbb{R}^{d_{uu}} \oplus \mathbb{R}^{d_c} \oplus \mathbb{R}^{d_s}$, where $d_u = d_{uu} + d_c$. In particular, A and B are products $A = A^{uu} \times A^c \times A^s$, $B = B^{uu} \times B^c$ and each B^u_j is a product $B^{uu}_j \times B^c_j$ where $B^c_j = A^c(B^c) + \pi^c(v_j)$. We thus obtain a family of affine

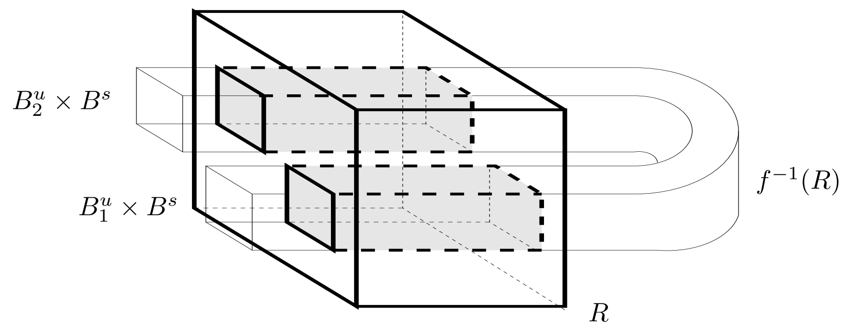

FIGURE 13. A standard affine horseshoe.

contractions L_j, $1 \le j \le \ell$, which send B^c to B_j^c respectively and coincide with $z \mapsto (A^c)^{-1}(z - \pi^c(v_j))$, where $\pi^c \colon \mathbb{R}^{d_u+d_s} \to \mathbb{R}^{d^c}$ is the projection onto the center coordinate. This defines a (finitely generated) iterated function system (IFS) in B^c.

Denoting $\beta = |\det(A^{uu})|$, we have that each strong unstable plaque $B^{uu} \times \{z\}$ with $z \in B^c \times B^s$ can intersect at most β distinct rectangles $B_j^u \times B^s$. This remark applied to any iterate f^n shows that for any point $x \in B^c$,

$$\operatorname{Card}\{(j_1, \dots, j_n) \in \{1, \dots, \ell\}^n,\ x \in L_{j_n} \circ \dots L_{j_1}(B^c)\} \le \beta^n. \tag{24}$$

We state a definition for IFS which is the analogue of Definition 7.1.

Definition 7.5. The IFS $\mathcal{L} = \{L_j \colon B^c \to B^c : j = 1, \dots, \ell\}$ satisfies the *recurrent compact condition* if there exists a nonempty compact set $K^c \subset \bigcup_j B_j^c$ such that for every $z \in K^c$ there exist $z' \in \text{interior}(K^c)$ and $1 \le j \le \ell$ with $L_j(z') = z$.

Proposition 7.6. *If the central IFS associated to a partially hyperbolic standard affine horseshoe Λ as above satisfies the recurrent compact condition, then Λ admits a recurrent compact set.*

Proof. We introduce the local strong unstable lamination by plaques of the form $[0,1]^{d_{uu}} \times \{a\}$ where $a \in [0,1]^{d_c+d_s}$. We then define the section $\mathcal{D} = \{1/2\}^{d_{uu}} \times (0,1)^{d_c+d_s}$ and introduce the compact set $K = K^c \times \bigcup_j B_j^s$.

Let us consider $x = (x_{uu}, x_c, x_s)$ in $\Lambda \subset \mathbb{R}^{d_{uu}} \times \mathbb{R}^{d_c} \times \mathbb{R}^{d_s}$ and a point $z = \{1/2\}^{d_{uu}} \times (z_c, x_s)$ in $W^u_{loc}(x) \cap K$. We have $z_c \in K^c$, hence there exists $z'_c \in \text{interior}(K^c)$ and $1 \le j \le \ell$ with $L_j(z'_c) = z$. There exists a point $x' \in \Lambda$ of the form $x' = (x'_{uu}, x'_c, x'_s)$ such that $A(x) + v_j$ belongs to $W^u_{loc}(x')$. The point $z' = \{1/2\}^{d_{uu}} \times (z'_c, x'_s)$ has the property that $f^{-1}(\overline{W^{uu}_{loc}(z')}) \subset W^{uu}_{loc}(z)$. Moreover since z'_c belongs to interior(K^c), the point z' belongs to interior(K) as required. $\square$

7.3. **Perturbations of iterated function systems.** Let $B = [-1/2, 1/2]^d$, let $L \in \mathrm{GL}(d, \mathbb{R})$ be a linear contraction, and write $J = |\det(L)|$. For $H \ge 1$, let $v_0, \dots, v_H \in B$ such that the affine contractions $L_j \colon z \mapsto L.z + v_j$ send B into its interior. For any finite word $\underline{j} = (j_1, ..., j_n)$ in $\{1, \dots, H\}^n$ let $L_{\underline{j}} = L_{j_n} \circ \dots \circ L_{j_1}$. For n large enough, we consider the IFS $\mathcal{L}_n = \{L_0 \circ L_{\underline{j}},\ \underline{j} \in \{1, \dots, H\}^n\}$.

We will assume furthermore that there exists $\beta > 1$ and $c \in (0,1)$ such that:

– $\beta^{2-c} < (JH)^2$,
– $\text{Card}\left\{\underline{j} \in \{1,\dots,H\}^n,\ x \in L_{\underline{j}}(B)\right\} \le \beta^n$ for any $n \ge 1$ and $x \in B$.

For $n \ge 1$ we define $m = [c \cdot n] + 1$ and consider the space Ω_n of functions $w\colon \{1,\dots,H\}^n \to B$ that satisfy $w_{\underline{j}} = w_{\underline{j}'}$ each time the last m letters $j_{n-m+1},\dots,j_n$ of $\underline{j}$ and $\underline{j}'$ coincide. Since Ω_n can be identified with B^{H^m} it is a probability space (endowed with the product Lebesgue measure).

The following probabilistic argument allows us to perturb the initial iterated function system $\mathcal{L}_n$ so that the recurrent compact condition is satisfied.

Proposition 7.7. *Under the previous assumptions, if n is large enough, then there exists $w \in \Omega_n$ such that the IFS induced by the affine contractions*

$$z \mapsto L_0 \circ L_{\underline{j}}(z) + 10.L^{n+1}(w_{\underline{j}}), \quad \underline{j} \in \{1,\dots,H\}^n$$

satisfies the recurrent compact condition.

Proof. Let $\alpha_n = \frac{1}{2}J^{n+1}(\beta^{-1}H)^n$.

Claim. *There exists a compact set $A \subset B$ with Lebesgue measure $|A| \ge \alpha_n$ of points x that belong to a least $\frac{1}{2}J^{n+1}H^n\beta^{m-n}$ images $L_0 \circ L_{\underline{j}}(B)$ associated to sequences $\underline{j} \in \{1,\dots,H\}^n$ whose m last letters $j_{n-m+1},\dots,j_n$ are different.*

Proof. Let $A \subset B$ be the set of points x that belong to at least $J^{n+1}H^n/2$ images $L_0 \circ L_{\underline{j}}(B)$ where $\underline{j} \in \{1,\dots,H\}^n$. By our assumptions, a point x can belong to at most β^n such images (here we apply the estimate (24) to the point $L_0^{-1}(x)$). Considering the images of B by all possible maps $L_0 \circ L_{\underline{j}}$, this gives:

$$\begin{aligned} J^{n+1}H^n = \sum_{\underline{j}} |L_0 \circ L_{\underline{j}}(B)| &= \int_B \text{Card}\{\underline{j} \in \{1,\dots,H\}^n, x \in L_0 \circ L_{\underline{j}}(B)\}dx \\ &\le \beta^n|A| + \frac{J^{n+1}H^n}{2}. \end{aligned}$$

This gives $|A| \ge \alpha_n$.

Fix $j_{n-m+1},\dots,j_n$ in $\{1,\dots,H\}^m$ and some $x \in B$. The number of images $L_0 \circ L_{\underline{j}}(B)$ containing x and whose last m letters of $\underline{j}$ coincide with $j_{n-m+1},\dots,j_n$ is equal to the number of images $L_{\underline{j}'}(B)$ with $\underline{j}' \in \{1,\dots,H\}^{n-m}$. It is thus smaller than β^{n-m}. It follows that the points $x \in A$ belong to at least $\frac{1}{2}J^{n+1}H^n\beta^{m-n}$ images $L_0 \circ L_{\underline{j}}(B)$ associated to sequences $\underline{j} \in \{1,\dots,H\}^n$ whose m last letters $j_{n-m+1},\dots,j_n$ are different. ◻

Consider the tiles $L^{n+1}(u+B)$, where $u \in \mathbb{Z}^d$, and their cores $L^{n+1}(u+\frac{1}{2}B)$. We denote by K the union of the tiles which intersect A and by $\tilde{K}$ the union of the cores of these tiles. We have $|\tilde{K}| = 2^{-d}|K| \ge 2^{-d}\alpha_n$. Finally, we define

$$A' = K \cap L^{n+1}\left(\frac{1}{100}.\mathbb{Z}^d\right).$$

Then the conclusion of the proposition holds for any parameter $w \in \Omega_n$ such that:

$$A' \subset \bigcup_{\underline{j} \in \{1,\dots,H\}^n} L_0 \circ L_{\underline{j}}(\tilde{K}) + 10.L^{n+1}(w_{\underline{j}}), \tag{25}$$

For a fixed $z \in A'$, we estimate the measure of the set of parameters

$$\Omega_n(z) := \{w \in \Omega_n,\ z \notin \bigcup_{\underline{j}\in\{1,\dots,H\}^n} L_0 \circ L_{\underline{j}}(\tilde{K}) + 10.L^{n+1}(w_{\underline{j}})\}. \tag{26}$$

Pick some $z_0 \in A$ which belongs to the tile as z. For each $\underline{j} \in \{1,\dots,H\}^n$ such that $z_0 \in L_0 \circ L_{\underline{j}}(B)$, the probability that $z \in L_0 \circ L_{\underline{j}}(\tilde{K}) + 10.L^{n+1}(w_{\underline{j}})$ is

$$\frac{|L_0 \circ L_{\underline{j}}(\tilde{K})|}{|10.L^{n+1}(B)|} = 10^{-d}|\tilde{K}| \geq 20^{-d}\alpha_n.$$

The point z belongs to at least $\frac{1}{2}J^{n+1}H^n\beta^{m-n}$ images $L_0 \circ L_{\underline{j}}(B)$ associated to words $\underline{j} \in \{1,\dots,H\}^n$ whose last m letters $j_{n-m+1},\dots,j_n$ are pairwise different. For these different words $\underline{j}$, the events $z \in L_0 \circ L_{\underline{j}}(\tilde{K}) + 10.L^{n+1}(w_{\underline{j}})$ are independant in the parameter space Ω_n. Consequently, the probability of the event (26) is at most $(1-(20^{-d}\alpha_n))^{\frac{1}{2}J^{n+1}H^n\beta^{m-n}}$. From the definition of α_n and m, this gives:

$$|\Omega_n(z)| \leq \exp\left(-100^{-d}J^{2(n+1)}H^{2n}\beta^{(c-2)n}\right).$$

By our choice of c, we have $J^2H^2\beta^{c-2} > 1$. Since the cardinality of A' grows exponentially with n, it follows that the measure of the set

$$\Omega_n \setminus \bigcup_{z\in A'} \Omega_n(z)$$

becomes arbitrarily small. In particular, for n large there exists $w \in \Omega_n$ such that condition (25) holds. Hence the result follows. □

7.4. **Reduction to standard affine horseshoes.** Any affine horseshoe contains large standard horseshoes (Definition 7.4). Note that the condition (5) can be preserved by this construction.

Proposition 7.8. *Consider f and an affine horseshoe Λ with constant linear part. Then for any $\varepsilon > 0$ there exist $\Lambda' \subset \Lambda$, a chart $\varphi\colon U \to \mathbb{R}^{d_u+d_s}$ and N such that*

- *$[0,1]^{d_u+d_s} \subset \varphi(U)$ and Λ' is a standard horseshoe of f^N for the chart φ,*
- *$\frac{1}{N}h_{top}(\Lambda', f^N) > h_{top}(\Lambda, f) - \varepsilon$,*
- *the diameter of $R := \varphi^{-1}([0,1]^{d_u+d_s})$ is smaller than ε,*
- *if $R_1,\dots,R_\ell$ are the connected components of $R \cap f^{-1}(R)$ that intersect Λ', then $f^i(R_j) \cap R = \emptyset$ for each $1 \leq i < N$ and $1 \leq j \leq \ell$.*

Proof. We will by extraction reduce to the case of a horseshoe whose dynamics are conjugate to a full shift.

Lemma 7.9. *Consider a diffeomorphism f, a horseshoe Λ_0 and $\varepsilon > 0$. Then there exist compact subsets $\Lambda_1 \subset \Lambda_0$ and integers $s, \ell \geq 1$ such that*

- *$\Lambda_1 = f^s(\Lambda_1)$ and $f^i(\Lambda_1) \cap \Lambda_1 = \emptyset$ for $1 \leq i < s$,*
- *the restriction of f^s to Λ_1 is conjugate to the shift on $\{0,\dots,\ell\}^{\mathbb{Z}}$, and*
- *$\frac{1}{s}\log(\ell) > h_{top}(\Lambda_0, f) - \varepsilon/2$.*

Proof. By [An2], any horseshoe is topologically conjugate to a transitive Markov subshift (X_0, σ_0) over a finite alphabet $\mathcal{A}_0$, hence it is enough to work in this setting.

We first claim that X_0 contains a compact invariant subset topologically conjugate to a transitive Markov subshift (X, σ) over another finite alphabet $\mathcal{A} = \{0, \dots, m\}$ such that $Z := X \cap \{1, \dots, m\}^{\mathbb{Z}}$ has topological entropy larger than

$$h_{top}(X, \sigma) > h_{top}(Z, \sigma) \geq h_{top}(X_0, \sigma_0) - \varepsilon/4. \tag{27}$$

To see this, fix a symbol $* \in \mathcal{A}_0$ and a sequence $(*, b_1, \dots, b_{p-1}, *)$, with $b_i \neq *$, that appears in X_0. There exists a multiple k of p such that X_0 contains a collection $\mathcal{W}$ of at least $\exp(k.(h_{top}(X, \sigma) - \varepsilon/4))$ different words $(*, a_1, a_2, \dots, a_{k-1})$ of length k such that

- the transition $(a_{k-1}, *)$ appears in X_0, and
- $(a_1, \dots, a_{p-1}) \neq (b_1, \dots, b_{p-1})$.

If $\mathcal{A}$ is the alphabet of words of length k over $\mathcal{A}_0$ that appear in X_0, then X_0 is topologically conjugate to a Markov subshift (X, σ) of $\mathcal{A}^{\mathbb{Z}}$ via the map that associates to any $(a_j) \in \{0, \dots, n\}^{\mathbb{Z}}$ the sequence of sub words $(a_j, a_{j+1}, \dots, a_{j+k-1})$ of length k.

Now fix the word w_0 of length k of X_0 obtained by concatenating the sequence $(*, b_1, \dots, b_{p-1})$. Label $\mathcal{A}$ by the integers $0, ..., m$ in such a way that the word w_0 corresponds to the symbol 0. Finally, consider the σ_0-invariant set $Z := X \cap \{1, ..., m\}^{\mathbb{Z}}$ of sequences $(a_j) \in X_0$ that do *not* contain the subword w_0. Note that Z contains all the sequences obtained by concatenating the words in the collection $\mathcal{W}$. In particular the topological entropy on Z is larger than $h_{top}(X, \sigma) - \varepsilon/2$. The claim follows.

We now continue the proof of the lemma, working with the subshift (X, σ). There exists $s \geq 1$ arbitrarily large such that X contains

$$\ell \geq \exp(s(h_{top}(Z, \sigma) - \varepsilon/4)) \tag{28}$$

different words of length $s+1$ of the form $0, a_1, b_2, \dots, a_{s-1}, 0$ with $a_j \neq 0$. We then define X' as the set of elements $(a_j)_{j \in \mathbb{Z}} \in X$ such that 0 appears exactely once in each subword of length s. If $X_1 \subset X'$ is the set of sequences $(a_j) \in \{0, \dots, m\}^{\mathbb{Z}} \cap X'$ such that $a_0 = 0$, then X' is the disjoint union of the iterates $\sigma^i(X_1)$, $0 \leq i < s$. Moreover, (X_1, σ^s) is conjugate to the full shift on ℓ symbols. From (27) and (28), the third item of the lemma follows. $\square$

From Lemma 7.9 it is enough to prove the proposition for f^s on Λ_1. Thus we may now assume that the dynamics of f on Λ is conjugate to the shift on $\{0, \dots, \ell\}^{\mathbb{Z}}$ by a homeomorphism h and that the entropy on the subhorseshoe $h^{-1}(\{1, \dots, \ell\}^{\mathbb{Z}})$ obtained by deleting one symbol is larger than $h_{top}(\Lambda, f) - \varepsilon/2$.

Since Λ is affine, up to a conjugacy, we may assume that it is contained in $\mathbb{R}^{d_u + d_s}$ and that f is piecewise affine in a neighborhood of Λ with a constant linear part $A = A^u \times A^s$. By an affine change of the coordinates, we may also assume that A^s sends $B^s = [0, 1]^{d_s}$ into its interior and that $(A^u)^{-1}$ sends sends $B^u = [0, 1]^{d_s}$ into its interior. Let $p \in \Lambda$ denote the fixed point $h^{-1}(\underline{0})$ and let R be a small cube centered at p that is the pre image of the standard cube $B^u \times B^s$ by an affine map of the form $\varphi \colon z \mapsto \lambda.z + v$. Hence R is a product $R = R^u \times R^s$.

One can find $x^s \in W^s(p)$ and $x^u \in W^u(p)$ contained in interior$(R) \cap \Lambda$ such that $f^{-i}(x^s), f^i(x^u) \notin R$ for $i \geq 1$. Denote by $W^s_{loc}(x^u)$ the connected component of $W^s(x^u) \cap R$ containing x^u: it is a cube of the plane $p + \{0\}^{d_u} \times \mathbb{R}^{d_s}$. If R has been chosen small enough, we may assume that $f^i(W^s_{loc}(x^u))$ is disjoint from R for any $i \geq 1$. In the same way, denoting by $W^u_{loc}(x^s)$ the connected component of

$W^u(x^s) \cap R$ containing x^s, we may assume that $f^{-i}(W^u_{loc}(x^s))$ is disjoint from R for any $i \geq 1$.

We now fix $n \geq 1$ and consider m much larger. Set $N = 2n+m$. Let $(a^u_j) = h(x^s)$ and $(a^s_j) = h^(x^u)$. To any sequence $b = (b_1, \dots, b_s) \in \{1, \dots, \ell\}^s$ we associate the point $x(b) \in \Lambda$ such that $c_j = h(x(b))$ satisfies:

- $c_j = a^u_j$ for $j < n$,
- $c_j = b_{j-n}$ for $n \leq i < n+m$,
- $c_j = a^s_{j-2n-m+1}$ for $n+m \leq j$.

By construction $x(b)$ and $f^N(x(b)$ belong to the interior of R, and the forward images of the connected component of $W^s(x(b)) \cap R$ containing $x(b)$ (resp. the backward images of the connected component of $W^u(f^N x(b)) \cap R$ containing $f^N(x(b)))$ are disjoint from the boundary of R. Since f is affine in a neighborhood of Λ, this implies that the connected component $R(b)$ of $R \cap f^{-N}(R)$ containing $x(b)$ is a set of the form $R^u(b) \times R^s$, where $R^u(b)$ is contained in the interior of R^u. Analogously, $f^N(R(b))$ is a set of the form $R^u \times R^s(b)$, where $R^s(b)$ is contained in the interior of R^s. Moreover the iterates $f^i(R(b))$ for $1 \leq i < N$ are disjoint from R.

The ℓ^n rectangles $R(b)$ (resp. $f^N(R(b)))$ for $b \in \{1, \dots, \ell\}^n$ are pairwise disjoint. Consequently, the maximal invariant set Λ' for f^N in $\bigcup_b R(b)$ is a standard horseshoe.

If n is sufficiently large, then the entropy of f^N on Λ' satisfies the required lower bound:

$$\frac{1}{N} h_{top}(\Lambda', f^N) = \frac{n}{N} \log(\ell) \geq \frac{n}{N}(h_{top}(\Lambda, f) - \varepsilon/2) \geq h_{top}(\Lambda, f) - \varepsilon.$$

□

7.5. **Proof of Theorem E.** We can now complete the construction of blenders for affine horseshoes with large entropy. Consider f and an affine horseshoe Λ with constant linear part A that is partially hyperbolic and satisfies condition (5). Denote by $\chi^u_{inf}(A)$ and $\chi^u_{max}(A)$ the smallest and the largest positive Lyapunov exponents of A; by (5), we can choose $c \in (0,1)$ such that

$$c.k.\chi^u_{max}(A) < \chi^u_{min}(A), \quad \text{and} \tag{29}$$

$$h_{top}(\Lambda, f) > \log|\det(A_{|E^u})| - \frac{c}{2}\chi^u_{max}(A). \tag{30}$$

By Proposition 7.8, we can assume that there exist $N \geq 1$, a decomposition

$$\Lambda = \Lambda' \cup f(\Lambda') \cup \dots \cup f^{N-1}(\Lambda'),$$

and a chart $\varphi \colon U \to \mathbb{R}^{d_u+d_s}$ such that:

- Λ' is a standard horseshoe for f^N in the neighborhood $R := \varphi^{-1}([0,1]^{d_u+d_s})$,
- Λ' is contained in finitely many components $R_0, \dots, R_H$ of $R \cap f^{-N}(R)$,
- $f^i(R_j) \cap R = \emptyset$ for any $0 \leq j \leq H$ and $1 \leq i < N$, and
- $\frac{1}{N}\log(1+H) = \frac{1}{N} h_{top}(\Lambda', f^N)$ is arbitrarily close to $h_{top}(\Lambda, f)$.

Consequently from (30) we have:

$$\log(H) - \log|\det(A^N_{|E^u})| > -\frac{c}{2}\chi^u_{max}(A^N) \geq -\frac{c}{2}\log|\det(A^N_{|E^{uu}})|. \tag{31}$$

As explained in Section 7.2, the standard horseshoe Λ' of f^N defines an IFS inside the center space E^c with affine contractions of the form $L_j \colon z \mapsto L.z + v_j$,

$0 \le i \le H$. Moreover $J := |\det(L)|$ coincides with $|\det(A^N_{E^u})|^{-1}$ and if we set $\beta = |\det(A^N_{|E^{uu}})|$, then the property (24) holds and (31) gives $\beta^{2-c} < (JH)^2$.

We can thus apply Proposition 7.7 and find $n \ge 1$ large and a function $w\colon \{1,\dots,H\} \to B^c$ such that, setting $m = [c.n]+1$, we have:

– the IFS induced by the affine contractions

$$z \mapsto L_0 \circ L_{\underline{j}}(z) + 10.L^{n+1}(w_{\underline{j}}), \quad \underline{j} \in \{1,\dots,H\}^n$$

satisfies the recurrent compact condition,

– we have $w_{\underline{j}} = w_{\underline{j}'}$ each time the last m letters of $\underline{j}$ and $\underline{j}'$ coincide.

After conjugating in the chart φ, the horseshoe Λ' for f^N is the maximal invariant set in the cube $[0,1]^{d_u+d_s} = B^u \times B^s$ for the dynamics induced by some affine maps:

$$F_j\colon B^u_j \times B^s \to B^u \times B^s_j\colon \qquad z \mapsto A^N(z) + v_j.$$

Consider the standard horseshoe Λ'' for $f^{(n+1).N}$ contained in Λ' and defined by the affine maps $F_{\underline{j}} \circ F_0$ where $\underline{j} \in \{1,\dots,N\}^N$ and $F_{\underline{j}} = F_{j_1} \circ F_{j_2} \circ \dots \circ F_{j_n}$. Each of these maps is defined on a domain $B^u_{(\underline{j},0)} \times B^s$. If T_j denotes the contraction of B^u induced by the map F_j^{-1}, then $B^u_{(\underline{j},0)} = T_0 \circ T_{j_n} \circ \dots \circ T_{j_1}(B^u)$.

We now explain how to modify f on $B^u_0 \times U^s$, where U^s is a small neighborhood of $B^s \subset \mathbb{R}^{d_s}$. Start by fixing a word $\underline{i} \in \{1,\dots,H\}^m$. Now consider the image $B^u_{\underline{i}} = T_0 \circ T_{i_m} \circ \dots \circ T_{i_1}(B^u)$; it contains all the sets $B^u_{(\underline{j},0)}$ such that the last m letters of $\underline{j}$ coincide with $\underline{i}$. Moreover the distance from each such set $B^u_{(\underline{j},0)}$ to the boundary of $B^u_{\underline{i}}$ is greater than $\delta.\exp(-\chi^u_{max}.m.N)$, where δ is a lower bound for the distance of the sets B^u_j to the complement of B^u.

By construction, there exists $w_{\underline{i}} \in B^c$ that coincides with the π^c-projection of all the vectors $w_{\underline{j}}$ associated to the words $\underline{j}$ whose last letters coincide with $\underline{i}$. We modify f inside $B^u_{\underline{i}} \times U^s$ by a diffeomorphism that coincides on each domain $B^u_{(\underline{j},0)} \times U^s + (0, 10.L^{(n+1)}(w_{\underline{i}}), 0)$ with

$$(z^{uu}, z^c, z^s) \mapsto f\big(z^{uu}, z^c - 10.L^{(n+1)}(w_{\underline{i}}), z^s\big).$$

Note that

$$\|10.L^{(n+1)}(w_{\underline{i}})\|^{1/k} \le 10\exp(-\chi^u_{min}.(n+1).N/k)$$

is much smaller than $\delta.\exp(-\chi^u_{max}.m.N)$, and hence than the distance between the sets $B^u_{\underline{j}}$ and the complement of $B^u_{\underline{i}}$, from inequality (29). Consequently diffeomorphism we obtain is C^k-close to f, provided n has been chosen large enough. Repeating this construction independently in each domain $B^u_{\underline{i}} \times U^s$, we obtain the diffeomorphism g. Note that g can be chosen to preserve the volume if f does.

By construction, the $(n+1).N$ consecutive iterates of $B^u_0 \times U^s$ by f are disjoint. It follows that the diffeomorphism $G = g^{(n+1).N}$ has a standard horseshoe Λ''_G whose IFS coincides with the affine contractions $z \mapsto L_0 \circ L_{\underline{j}}(z) + 10.L^{n+1}(w_{\underline{j}})$, $\underline{j} \in \{1,\dots,H\}^n$. By Propositions 7.3 and 7.6 this horseshoe is a d_{cs}-stable blender. Consequently the union of the iterates $g^i(\Lambda''_G)$, $1 \le i \le (n+1).N$, is a d_{cs}-stable blender for g, as required. Note also that the support of the perturbation is contained in a small neighborhood of R, which has arbitrarily small diameter. $\square$

8. Appendix: Approximation of hyperbolic measures by horseshoes

We prove in this section the version of Katok's approximation of hyperbolic measures stated in Theorem 1.5. Recall that we have fixed a diffeomorphism f in $\mathrm{Diff}^{1+\alpha}(M)$, an ergodic hyperbolic measure μ, a constant $\delta > 0$ and a neighborhood $\mathcal{V}$ of μ in the weak* topology.

8.1. **Uniformity blocks.**

Our goal is to extract a subset of M that generates a horseshoe with large entropy. In order to select enough orbit types, we will need to shadow certain pseudo-orbits for f by true f-orbits. Such shadowing lemmas exist for nonuniformly hyperbolic dynamics, but we need one especially adapted to our setting, allowing us to control Lyapunov exponents. The orbits will be selected from points visiting a special set called a *uniformity block*.

We first introduce the Oseledets-Pesin charts associated to an ergodic measure. See Theorem S.3.1 in [KH].

Theorem 8.1 (Pesin). *Let $f\colon M \to M$ be a $C^{1+\alpha}$ diffeomorphism preserving the ergodic probability measure μ and let $\eta > 0$. Then there exist $Z \subset M$, measurable with $\mu(Z) = 1$, two measurable functions $r, K\colon M \to (0, 1]$ and a measurable family of invertible linear maps $C(x)\colon T_xM \to \mathbb{R}^d$ satisfying the following properties.*

(1) $|\log(r(f(x))/r(x))| < \eta$ *and* $|\log(K(f(x))/K(x))| < \eta$ *for any* $x \in Z$.

(2) *If $\chi_1 > \cdots > \chi_\ell$ are the Lyapunov exponents of μ, with multiplicities $n_1, \dots, n_\ell$, then for any $x \in Z$ there are $A_i(x) \in GL(n_i, \mathbb{R})$ such that*
$$L = C(f(x)) \cdot Df(x) \cdot C^{-1}(x) = \mathrm{diag}\left(A_1(x), \dots, A_\ell(x)\right),$$
$$e^{\chi_i - \eta} < \|A_i^{-1}\|^{-1} \le \|A_i\| < e^{\chi_i + \eta}.$$

(3) *In the charts $\phi_x\colon x \mapsto C(x) \circ \exp_x^{-1}$, the maps $f_x = \phi_{f(x)} \circ f \circ \phi_x^{-1}$ are defined on $B(0, r(x))$ and satisfy $d_{C^1}(f_x, Df_x(0)) < \eta$ at every $x \in Z$.*

(4) *For any $x \in Z$ and any $y, y' \in \phi_x^{-1}(B(0, r(x)))$,*
$$d(y, y') \le \|\phi_x(y) - \phi_x(y')\| \le K^{-1}(x)\, d(y, y').$$

The sets $\phi_x^{-1}(B(0, r(x)))$ are called *regular neighborhoods* of the points $x \in Z$. They decay slowly exponentially in size (at rate at most $e^{-\eta}$) along the orbits.

Definition 8.2. A compact set $X \subset M$ with $\mu(X) > 0$ is a *uniformity block for μ (with tolerance $\eta > 0$)* if there exist Z, r, K, C as in Theorem 8.1 with $X \subset Z$ such that r, K, C are continuous on X.

For any tolerance $\eta > 0$, Theorem 8.1 and Lusin's theorem imply the existence of uniformity blocks with measure arbitrarily close to 1.

8.2. **Shadowing theorem.**

The shadowing theorem we will use applies to special pseudo-orbits – those with jumps in a uniformity block X for a hyperbolic measure μ. A sequence $(x_n)_{n\in\mathbb{Z}} \subset M$ is an *ϵ-pseudo-orbit with jumps in a set $X \subset M$* if $\{x_n, n \ge n_0\}$ and $\{x_n, n \le -n_0\}$ meet X for arbitrarily large n_0 and moreover
$$\forall n \in \mathbb{Z}, \quad d(f(x_n), x_{n+1}) > 0 \implies f(x_n), x_{n+1} \in X \text{ and } d(f(x_n), x_{n+1}) < \epsilon.$$
The following theorem uses the hyperbolicity assumption on the measure μ.

Theorem 8.3. *For any $f \in \mathrm{Diff}^{1+\alpha}(M)$ with ergodic hyperbolic measure μ, there exists $\kappa > 0$ such that, for every $\eta > 0$ sufficiently small and every uniformity block X for μ of tolerance η, the following property holds for some constants $C_0, \epsilon_0 > 0$.*

If (x_n) is an ϵ-pseudo-orbit with jumps in X and $\epsilon \in (0, \epsilon_0)$, then there exists a unique $y \in M$ whose orbit $C_0\epsilon$-shadows (x_n), i.e. $d(f^n(y), x_n) < C_0\epsilon$ for all $n \in \mathbb{Z}$. Moreover y belongs to the regular neighborhood $\phi_x^{-1}(B(0, r(x)/2))$.

If y, y' shadow two pseudo-orbits (x_n), (x'_n) such that $x_n = x'_n$ for $|n| \leq N$, then

$$d(y, y') \leq C_0 e^{-\kappa N}.$$

We will use the classical shadowing theorem for sequences of diffeomorphisms (see, e.g. [Pi]).

Theorem 8.4. *For every $\kappa > 0$ there exist $\theta, \eta_0 > 1$ with the following property.*

Let $(L^s_n) \in \mathrm{GL}(d_s, \mathbb{R})^{\mathbb{Z}}$ and $(L^u_n) \in \mathrm{GL}(d_u, \mathbb{R})^{\mathbb{Z}}$ satisfying $\|L^s_n\|, \|(L^u_n)^{-1}\| < e^{-\kappa}$ and for each n let $L_n = \mathrm{diag}(L^u_n, L^s_n)$. Let $(g_n) \in \mathrm{Diff}^1(\mathbb{R}^d)^{\mathbb{Z}}$, where $d = d_u + d_s$, be a sequence of diffeomorphisms with $d_{C^1}(g_n, L_n) < \eta_0$ and fix any $\epsilon > 0$.

Then every ϵ-pseudo-orbit for (g_n) is $\theta\epsilon$-shadowed by a unique (g_n)-orbit. More precisely: for every sequence (x_n) in $\mathbb{R}^d$ satisfying $\|g_n(x_n) - x_{n+1}\| < \epsilon$ for all $n \in \mathbb{Z}$, there exists a sequence (y_n) in $\mathbb{R}^d$ satisfying $g_n(y_n) = y_{n+1}$ and $\|y_n - x_n\| < \theta\epsilon$ for all $n \in \mathbb{Z}$. This sequence is unique: for any sequence (z_n) in $\mathbb{R}^d$ satisfying $g_n(z_n) = z_{n+1}$ for all $n \in \mathbb{Z}$, and for any $n_1 \leq k \leq n_2$, we have:

$$\|z_k - y_k\| \leq e^{-\kappa(k-n_1)}\|z_{n_1} - y_{n_1}\| + e^{-\kappa(n_2-k)}\|z_{n_2} - y_{n_2}\|.$$

Proof of Theorem 8.3. Choose a positive lower bound χ for the $|\chi_i|$. We set $\kappa = \chi/2$ and get constants θ, η_0 from Theorem 8.4. We choose $\eta < \min(\chi/2, \eta_0/4)$. We consider a uniformity block X for μ with tolerance η and associated to functions r, K, C. Assuming that η is small enough, for any C^1 map $f_0 \colon B(0, r) \to \mathbb{R}^d$ and for any linear map $L \in \mathrm{GL}(d, \mathbb{R})$ such that $d_{C^1}(f_0, L) < \eta$, there exists a C^1-map $g_0 \colon \mathbb{R}^d \to \mathbb{R}^d$ which coincides with f_0 on $B(0, r/2)$ and satisfies $d_{C^1}(g_0, L) < 2\eta$. We extend in this way each diffeomorphism f_x to a diffeomorphism g_x of $\mathbb{R}^d$ agreeing with f_x on $B(0, r(x)/2)$ and such that $d_{C^1}(g_x, Df_x(0)) < 3\eta$.

Let $\Delta > 1$ be an upper bound for K^{-1}, $x \in X$. We choose $\epsilon_0 > 0$ such that $r(x) > 2\theta\Delta\epsilon_0$ for each $x \in X$. Moreover if ϵ_0 is small enough, when $f(x), x' \in X$ are ϵ_0-close, the diffeomorphism $f_{x,x'} = \phi_{x'} \circ f \circ \phi_x^{-1}$ and f_x are η-close in the C^1-topology. In particular $f_{x,x'}$ can be extended to a diffeomorphism $g_{x,x'}$ of $\mathbb{R}^d$ that coincides with $f_{x,x'}$ on $B(0, r(x)/2)$ so that $d_{C^1}(g_{x,x'}, Df_x(0)) < 3\eta$.

Fix $\epsilon < \epsilon_0$ and let (x_n) be a ϵ-pseudo-orbit with jumps in X. We consider the sequence (g_n) of diffeomorphisms of $\mathbb{R}^d$ defined by $g_n = g_{x_n}$ if $x_{n+1} = f(x_n)$ and $g_n = g_{x_n, x_{n+1}}$ otherwise. In the first case $g_n(0) = 0$ and in the second case $|g_n(0)| \leq \Delta\epsilon$. The pseudo-orbit (0) is thus $\theta\Delta\epsilon$-shadowed by an orbit $(\overline{y}_n)$ of (g_n).

At times n such that $x_n \in X$ we have $\overline{y}_n \in B(0, r(x_n)/2)$ by our choice of ϵ_0. At other times, we consider the smallest interval $\{n_1, \dots, n_2\}$ of $\mathbb{Z}$ containing n such that $x_{n_1}, x_{n_2} \in X$. The sequence $(\overline{y}_n)$ may be compared to the orbit of (g_n) which coincides with 0 for the indices $n_1, \dots, n_2 - 1$. Consequently,

$$\|\overline{y}_n\| \leq (e^{-\kappa(n-n_1)} \ + \ e^{-\kappa(n_2-k)})\theta\Delta\epsilon.$$

From the property (1) in Theorem 8.1 we have:

$$r(x_n) \geq \max(e^{-\eta(n-n_1)} r(x_{n_1}) \ , \ e^{-\eta(n_2-n)} r(x_{n_2})),$$

which implies that $\overline{y}_n \in B(0, r(x_n)/2)$ still holds. The projection $y_n = \phi_{x_n}^{-1}(\overline{y}_n)$ satisfies $d(y_n, x_n) \leq d(\overline{y}_n, 0) \leq \theta\Delta\epsilon$, so that the sequence (y_n) $C_0\epsilon$-shadows (x_n) where $C_0 := \theta\Delta$. By construction (y_n) is an orbit of f.

If (y'_n) is another orbit that shadows a pseudo-orbit (x'_n) such that $x_n = x'_n$ for $|n| \leq N$, the lifts $\phi_{x_n}(y_n)$ and $\phi_{x_n}(y'_n)$ may be compared. From Theorem 8.4,

$$\begin{aligned} d(y_0, y'_0) &\leq \|\phi_{x_0}(y_0), \phi_{x_0}(y'_0))\| \\ &\leq 2e^{-\kappa N} \max(\|\phi_{x_N}(y_N), \phi_{x_N}(y'_N)\|), \|\phi_{x_{-N}}(y_{-N}), \phi_{x_{-N}}(y'_{-N}))\| \\ &\leq 2\theta\Delta e^{-\kappa N} \leq C_0 e^{-\kappa N}. \end{aligned}$$

This completes the proof of Theorem 8.3. □

8.3. **Metric entropy.** For $n \geq 1$ we define the dynamical distance

$$d_f(x, y) = \min\{d(f^k(x), f^k(y)),\ k = 0, \dots, n-1\}.$$

For $\rho > 0$ and $\beta > 0$, let $C_\mu(n, \rho, \beta)$ denote the minimal cardinality of the families of $d_{f,n}$-balls of radius ρ whose union $\cup_i B_{f,n}(x_i, \rho)$ has measure larger than β.

Theorem 8.5 (Katok, Theorem 1.1 in [Ka]). *For any $\beta > 0$,*

$$h(\mu, f) = \lim_{\rho\to 0} \varliminf_{n\to\infty} \frac{1}{n} \log C(n, \rho, \beta) = \lim_{\rho\to 0} \varlimsup_{n\to\infty} \frac{1}{n} \log C(n, \rho, \beta).$$

8.4. **Proof of Theorem 1.5.** We choose a $\gamma > 0$ and a finite collection of continuous functions $\psi_1, \dots, \psi_k$ on M such that $\mathcal{V}$ contains the set of probability measures ν satisfying:

$$\max_{1\leq i\leq k} \left| \int_M \psi_i \, d\mu - \int_M \psi_i \, d\nu \right| < \gamma.$$

The tolerance η. Define cones $\mathcal{C}^1, \dots, \mathcal{C}^{\ell-1}$ of $\mathbb{R}^d$ by:

$$\mathcal{C}^i = \{v = (v_1, v_2) \in \mathbb{R}^{d_i} \times \mathbb{R}^{d-d_i}\ \|v_1\| > \|v_2\},$$

where $d(i) = n_1 + \cdots + n_i$. Let $\eta \in (0, \min \|\chi_i\|)$ such that the shadowing theorem holds. We will consider linear maps $D \in \mathrm{GL}(d, \mathbb{R})$ that are η-close to diagonal maps $\mathrm{diag}(A_1, \dots, A_\ell)$ with

$$\exp(\chi_i - \eta) < \|A_i^{-1}\|^{-1} \leq \|A_i\| < \exp(\chi_i + \eta).$$

If η is chosen sufficiently small, then for any $i \in \{1, \dots, \ell-1\}$ and for any such D, the closure of the cone $\mathcal{C}^i$ is mapped by D inside $\mathcal{C}^i \setminus \{0\}$. Moreover if (v'_1, v'_2) is the image of $(v_1, v_2) \in \mathcal{C}^i$ by D, then

$$\exp(\chi_1 + \delta)\|v_1\| \geq \|v'_1\| \geq \exp(\chi_i - \delta)\|v_1\|.$$

Symmetrically, if $(v'_1, v'_2) \in \mathbb{R}^d \setminus \mathcal{C}^1$ is the image of (v_1, v_2) by D then

$$\exp(\chi_{i+1} + \delta)\|v_2\| \geq \|v'_2\| \leq \exp(\chi_d - \delta)\|v_2\|.$$

We then choose a uniformity block for μ with tolerance η.

The separation scale ρ. Choose $\xi > 0$ such that

$$\xi < \frac{\delta}{h(\mu, f) + 4} \quad \left(< \frac{\delta}{4}\right).$$

Next choose a separation scale $\rho \in (0, \delta)$ such that

$$d(x, y) < \rho \implies \max_{1 \leq i \leq k} |\psi_i(x) - \psi_i(y)| < \gamma/2,$$

and a separation time $N_0 \in \mathbb{N}$ such that $n \geq N_0$ implies

$$\frac{1}{n} \log C(n, \rho, \mu(X)/2) > h(\mu, f) - \xi.$$

For any time $n \geq N_0$, any subset of X of μ-measure at least $\mu(X)/2$ contains at least $\exp(n(h(\mu, f) - \xi))$ points whose orbit separate a distance ρ before time n.

The shadowing scale ϵ and the common return time N. We choose $\epsilon > 0$ so that $C_0\epsilon < \rho/2$, where C_0 is the shadowing scaling factor given by Proposition 8.3 for the uniformity block X.

Lemma 8.6. *There are $N \geq N_0$, an $\epsilon/2$-ball B centered at a point in X and a set $Y \subset B \cap X$ such that:*

- *the points of Y are ρ-separated in the distance $d_{f,N}$,*
- *$f^N(Y) \subset B \cap X$,*
- *$\left|\frac{1}{N} \sum_{j=1}^{N} \psi_i(f^j(x)) - \int_M \psi_i \, d\mu\right| < \gamma/2$ for each $y \in Y$ and $i \in \{1, \dots, k\}$, and*
- *the cardinality of Y is larger than* $\exp(N(h(\mu, f) - \delta))$.

Proof. Cover X by finitely many $\epsilon/2$-balls centered at points $x_1, \dots, x_t \in X$. For $m \geq N_0$, let

$$X_m^0 = \{x \in X : \exists n, i, \text{ s.t. } m \leq n < (1 + \xi)m,\ 1 \leq i \leq t, \text{ and } x, f^n(x) \in B(x_i, \epsilon/2)\},$$

$$\text{and } X_m = \left\{ x \in X_m^0 : \sup_{n \geq m} \max_{1 \leq i \leq k} \left| \frac{1}{n} \sum_{j=1}^{n} \psi_i(f^j(x)) - \int_M \psi_i \, d\mu \right| < \gamma/2. \right\}.$$

The Birkhoff Ergodic Theorem implies that $\mu(X \setminus X_m) \to 0$ as $m \to \infty$. Ffix $m > \max(N_0, \xi^{-1} \log t)$ such that $\mu(X_m) > \mu(X)/2$ for each $m > N_1$, and let E_m be a maximal (m, ρ)-separated set in X_m. By our choice of N_0, we have

$$\#E_m \geq \exp(m(h(\mu, f) - \xi)).$$

For $n \in \{m, (1 + \xi)m - 1\}$, let

$$V_n = \{x \in E_m : x, f^n(x) \in B(x_i, \epsilon/2), \text{ for some } i \in \{1, \dots t\}\},$$

and let N be a value of n maximizing $\#V_n$. Then

$$\#V_N \geq \frac{\#E_m}{\xi m} \geq \frac{e^{m(h(\mu,f)-\xi)}}{\xi m} > \exp(m(h(\mu, f) - 2\xi)).$$

Next choose $i \in \{1, \dots, t\}$ such that $B(x_i, \epsilon/2) \cap V_N$ has maximal cardinality, and let $B = B(x_i, \epsilon/2)$ and $Y = B(x_i, \epsilon/2) \cap V_N$. Then from the choice of ξ and since $m > \xi^{-1} \log(t)$, we get the required estimate

$$\#Y \geq \frac{\#V_N}{t} \geq \frac{1}{t} \exp(m(h(\mu, f) - 2\xi)) > \exp(N(h(\mu, f) - \delta)).$$

□

The set Λ. Consider the shift $\mathcal{Y} = Y^{\mathbb{Z}}$ over the alphabet Y. Since the diameter of B is smaller than ϵ, each $(y_n) \in \mathcal{Y}$ is associated to the ϵ-pseudo-orbit:

$$\dots, y_0, \dots, f^{N-1}(y_0), y_1, \dots, f^{N-1}(y_1), \dots,$$

which is obtained by concatenating the orbit segments of length N starting at the points y_n. Theorem 8.3 implies that for each such pseudo-orbit there is a unique point $\pi((y_n))$ whose orbit shadows, and which depends continuously on the sequence (y_n) (in the natural product topology on $\mathcal{Y} = Y^{\mathbb{Z}}$). Consequently, the union Λ_0 of all the orbits that shadow the elements of $\mathcal{Y}$ is an f^N-invariant compact set. By the last property of the shadowing theorem, the map $\pi\colon \mathcal{Y} \to \Lambda_0$ is continuous. Hence π conjugates the shift to f^N.

Since for distinct $y, y' \in Y$, there exists $j \in [0, N-1]$ such that $d(f^j(y), f^j(y')) > \rho$, and since $C_0\epsilon < \rho/2$, it follows that if x and x' $C_0\epsilon$-shadow distinct elements of $\mathcal{Y}$, then x and x' are distinct as well. This implies that π is a homeomorphism. We define $\Lambda = \Lambda_0 \cup f(\Lambda_0) \cup \dots \cup f^{N-1}(\Lambda_0)$. It is a transitive f-invariant compact set. From this we will obtain the entropy estimate (conclusion (2) of Theorem 1.5):

$$h(\Lambda, f) = \frac{1}{N} h(\Lambda_0, f^N) > h(\mu, f) - \delta.$$

From the construction of Y, for any $y_i \in Y$, we have

$$\max_{1 \le i \le k} \left| \frac{1}{N} \sum_{j=1}^{N} \psi_i(f^j(y_i)) - \int_M \psi_i \, d\mu \right| < \gamma/2.$$

By our choice of ρ, if for some $x \in X$, we have $\max_{0<j\le N} d(f^j(x), f^j(y_i)) < \rho$, then

$$\max_{1 \le i \le k} \left| \frac{1}{N} \sum_{j=1}^{N} \psi_i(f^j(y_i)) - \frac{1}{N} \sum_{j=1}^{N} \psi_i(f^j(x)) \right| < \gamma/2.$$

Since $C_0\epsilon < \rho$, it follows that if a point x $C_0\epsilon$-shadows a pseudo-orbit in $\mathcal{Y}$, then

$$\overline{\lim_{n\to\infty}} \max_{1 \le i \le k} \left| \frac{1}{n} \sum_{j=0}^{n-1} \psi_i(f^j(x)) - \int_M \psi_i \, d\mu \right| < \gamma.$$

From this conclusion (3) follows.

By construction Λ is contained in the $C_0\epsilon$-neighborhood of the support of μ. Assuming that $\mathcal{V}$ has been chosen small enough, Λ intersects any δ-ball centered at points in the support of μ, hence Λ is δ-close to the support of μ in the Hausdorff distance, which gives conclusion (1).

By the shadowing theorem, any orbit in Λ_0 stays in the regular neighborhoods of the pseudo-orbit it shadows. Moreover Λ_0 is contained in the regular neighborhood of the center x of B and is preserved by f^N. By properties (2) and (3) in Theorem 8.1 and by the choice of the tolerance constant, it follows that after conjugacy by the chart ϕ_x, the derivative of f^N preserves the cones $\mathcal{C}^1, \dots, \mathcal{C}^\ell$. The cone field criterion implies that Λ has a dominated splitting as in condition (4):

$$T_\Lambda M = E_1 \oplus \dots \oplus E_\ell, \quad \text{with } \dim(E_i) = n_i.$$

By definition and invariance of the cones, the growth of the iterates of any vector in E_i is given by its second coordinate in $\mathbb{R}^d = \mathbb{R}^{d(i-1)} \times \mathbb{R}^{n_i} \times \mathbb{R}^{d-d(i)}$. Consequently for v in E_i and $n \geq 1$ large enough we obtain condition (5):

$$\exp((\chi_i - \delta)n) \leq \|Df^n(v)\| \leq \exp((\chi_i + \delta)n).$$

Since the exponents χ_i are all different from zero, the set Λ is hyperbolic.

We can reduce to the case where Λ is a horseshoe by applying the following proposition.

Proposition 8.7. *Let K be a transitive hyperbolic set for a C^1-diffeomorphism f. Then for any $\delta > 0$, there exists a horseshoe Λ that is δ-close to K in the Hausdorff topology and satisfies $h_{top}(\Lambda, f) \geq h_{top}(K, f) - \delta$.*

Proof. Let $h = h_{top}(K, f)$. The shadowing lemma associates arbitrarily small constants $\epsilon, \rho > 0$ such that any ϵ-pseudo-orbit contained in K is ρ-shadowed by an orbit of f. Arguing as previously, there exists $N \geq 1$ arbitrarily large and a point $x \in K$ such that K contains a set $\mathcal{L}$ of sequences $L = (x_1, \dots, x_{N-1})$ such that:

- $\mathcal{L}$ contains at least $\exp(N(h - \delta/4))$ elements,
- each sequence $(x, L, x) = (x, x_1, \dots, x_{N-1}, x)$ is a ϵ-pseudo-orbit, and
- any two such sequences $L = (x_i)$, $L' = (x'_i)$ are 4ρ-separated: there exists $1 \leq i < N$ such that $d(x_i, x'_i) > \rho$.

We may assume that there is $L_0 = (x_1^0, \dots, x_{N-1}^0)$ in $\mathcal{L}$ such that the N-periodic pseudo-orbit $(x, L_0) = (x, x_1^0, \dots, x_{N-1}^0)$ is shadowed by an orbit whose minimal period is exactly N. Indeed the collection of periodic orbits which have same period and shadow different (x, L), $L \in \mathcal{L}$ are ρ-separated. Hence the number of those whose minimal period is smaller than $N/2$ is at most $\exp((h-\delta/4)N/2)$. This is less than the total number of elements of $\mathcal{L}$.

Fix such a sequence L_0. By construction the only sub-intervals of length $N-1$ of $(L_0, x, L_0) = (x_1^0, \dots, x_{N-1}^0, x, x_1^0, \dots, x_{N-1}^0)$ that are 2ρ-shadowed by L_0 are the initial and the final sub-intervals.

The number of sequences $L \in \mathcal{L}$ which 2ρ-shadow a sub-interval of length $N-1$ of (L_0, x, L_0) is smaller than N, by the separation assumption. Consequently, one can find a subset $\mathcal{L}' \subset \mathcal{L}$ with $\#\mathcal{L}' \geq \exp(N(h - \delta/2))$ such that the $L \in \mathcal{L}'$ do not ρ-shadow any sub-interval of length $N-1$ of (L_0, x, L_0).

We then choose ℓ large and considers pseudo-orbits of the form

$$\dots, x, L(0), x, L(1), x, L(2), x, \dots$$

such that $L(i) = L_0$ when $i = 0$ or $1 \bmod \ell$ and $L(i) \in \mathcal{L}'$ otherwise. Each of these sequences is shadowed by the orbit of a unique point. Their set is a compact set Λ_0 invariant by $f^N \ell$ and conjugate to the shift on $(\mathcal{L}')^{N(\ell-2)}$.

By the choice of L_0 and $\mathcal{L}'$, two different pseudo-orbits are 2ρ-separated and are ρ-shadowed by disjoint orbits. Consequently, Λ_0 is disjoint from its $N\ell - 1$ first iterates. The invariant compact set $\Lambda = \Lambda_0 \cup \dots \cup f^{N\ell-1}(\Lambda_0)$ is a hyperbolic set with entropy larger than $h - \delta$ and conjugate to a transitive subshift of finite type. By [An2]), Λ is a horseshoe. By construction it is arbitrarily close to K in the Hausdorff topology. $\square$

The proof of Theorem 1.5 is now complete.

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